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PROBABILISTIC ANALYSIS OF CRACKED FRAME STRUCTURES

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Key words: Stress intensity factors, Three-dimensional Frame Structure, Edge crack.

Parole chiave: Fattori di Intensificazione degli Sforzi, Struttura a Telaio Tridimensionali, Fessura.

Abstract. *In this paper, a numerical study on the structural behaviour of three-dimensional cracked structures is presented. The compliance matrix of the cracked element is given by the sum of the compliance matrix of the intact element and an additional compliance matrix which contains all the flexibilities given by the presence of the crack. Crack depth and location are modelled as random variables in order to take into account the unavoidable uncertainty that always affects damaged structures. A simple and accurate method for the probabilistic characterization of the linear elastic response of cracked 3D frame structures with uncertain damage is developed.*

Sommario. *In questo lavoro, viene presentato uno studio numerico sul comportamento strutturale di telai tridimensionali fessurati. La matrice di cedibilità dell'elemento fessurato è data dalla somma della matrice di cedibilità dell'elemento integro e di una matrice aggiuntiva che contiene tutte le cedibilità date dalla presenza della fessura. La profondità e la posizione della fessura sono modellate come variabili aleatorie. Viene impiegato un metodo semplice e affidabile per la caratterizzazione probabilistica della risposta elastica lineare di strutture intelaiate 3D con danno incerto.*

1 INTRODUCTION

In this study, three dimensional structures with cracked elements are considered. Crack depth and crack location are modelled as random variables in order to take into account the unavoidable uncertainty that always affects damaged structures. In the literature, the most common procedures for the stochastic analysis of structures with uncertain parameters are Monte Carlo simulation (see, for example, the survey paper[1]) and perturbation techniques (see, for example, the survey paper[2]). The main drawbacks of these approaches are, for the former, the high computational cost involved to obtain statistical convergence and, for the latter, the low accuracy as the level of uncertainty increases. Based on the above remarks, a computationally efficient and accurate method has been presented by Di Paola[3] to analyse truss structures with uncertain geometrical and mechanical properties. This approach has been generalized for the probabilistic analysis of linear elastic edge-cracked truss and frame structures with uncertain crack features in the two dimensional space[4,5]. In this paper, the stochastic method is applied to three-dimensional multicracked frame structures[6] aiming at assessing the overall reliability. Numerical results show the excellent performance of the approach to characterize accurately the structural response.

2 MODEL OF THE CRACKED BEAM IN 3D

Consider a linear elastic structure subjected to deterministic static loads. The response of the generic a-th structural element is governed by the following equations:

$$\mathbf{e}_a = \mathbf{D}_a \mathbf{u}_a \;\; \text{(compatibility)} \;,\; \mathbf{D}_a^{\mathrm{T}} \mathbf{q}_a = \mathbf{S}_a \;\; \text{(equilibrium)} \;,\; \mathbf{e}_a = \mathbf{C}_a \mathbf{q}_a \;\; \text{(constitutive)} \tag{1}$$

where $\mathbf{u}_a$ is the vector of nodal displacements, $\mathbf{e}_a$ is the vector of element deformations or generalized strains, $\mathbf{S}_a$ is the vector of nodal forces, $\mathbf{q}_a$ is the vector of element internal forces or generalized stresses (work conjugate to $\mathbf{e}_a$), $\mathbf{D}_a$ and $\mathbf{D}_a^{\mathrm{T}}$ are the compatibility and equilibrium matrices and $\mathbf{C}_a$ is the compliance matrix. For the sake of simplicity, distributed loads over the element are not considered. In the following we refer to Timoshenko beam-type elements. In the three-dimensional setting, the generalized nodal displacement and force vectors $\mathbf{u}_a$ and $\mathbf{S}_a$, respectively, of a beam element of length l are represented in Fig. 1. The element deformation components are collected in the vector: $\mathbf{e}_a^{\mathrm{T}} = \begin{bmatrix} \varphi_x & \varphi_{y1} & \varphi_{y2} & \varphi_{z1} & \varphi_{z2} & e_a \end{bmatrix}$, where φ_x is the twist deformation, φ_{y1} is the bending curvature in the plane x-z at node 1, φ_{y2} is the bending curvature in the plane x-z at node 2, φ_{z1} is the bending curvature in the plane x-y at node 1, φ_{z2} is the bending curvature in the plane x-y at node 2 and e_a is the axial elongation. The internal forces, work conjugate to the element deformations are defined by the vector $\mathbf{q}_a^{\mathrm{T}} = \begin{bmatrix} m_x & m_{y1} & m_{y2} & m_{z1} & m_{z2} & N \end{bmatrix}$. The classical relation between nodal forces and nodal displacements is obtained: $\mathbf{S}_a = \mathbf{k}_a \mathbf{u}_a$, with $\mathbf{k}_a = \mathbf{D}_a^{\mathrm{T}} \mathbf{C}_a^{-1} \mathbf{D}_a$ represents the element stiffness matrix. Now, let us consider an intact, homogeneous beam with constant cross-section. The compliance matrix for the intact beam is referred as $\mathbf{C}_a^{in}$. With the choice made for the components of the internal forces and deformations, it is easy to verify that the constitutive matrix, that is the inverse of the compliance matrix, has the form:

$$\left(\mathbf{C}_a^{in}\right)^{-1} = \begin{bmatrix} k_t & 0 & 0 & 0 \\ & \mathbf{k}_y & \mathbf{0} & 0 \\ & & \mathbf{k}_z & 0 \\ sym & & & k_a \end{bmatrix} \tag{2}$$

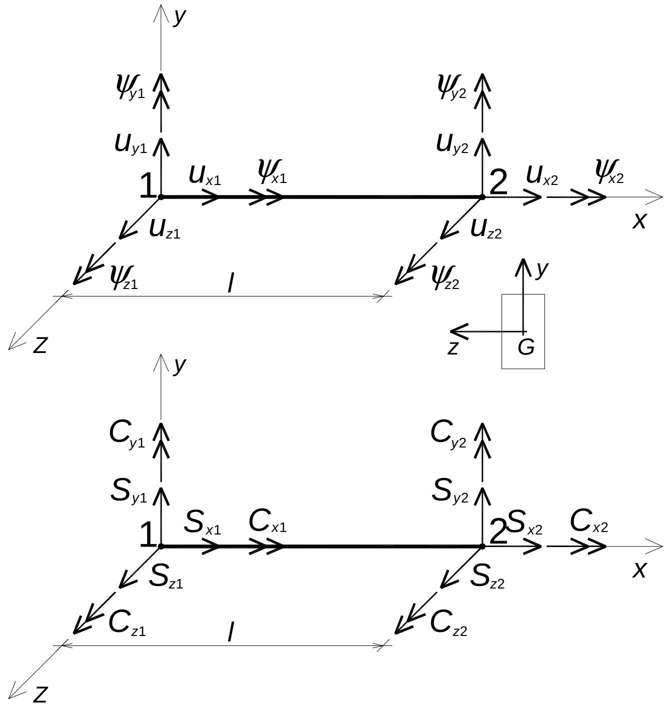

Figure 1: Three-dimensional Timoshenko beam element: nodal displacements and forces

where $k_t = GJ_t/l$ is the torsional stiffness of the beam with G shear modulus, J_t torsion constant, $\mathbf{k}_y$ is the stiffness matrix in the x-z plane, $\mathbf{k}_z$ is the stiffness matrix in the x-y plane and $k_a=EA/l$ is the axial stiffness with E elastic modulus and A cross-sectional area.

Consider now a cracked beam with rectangular cross-section Bxh as represented in Fig. 2. The beam has an edge crack of depth $a = \alpha$ x h, with α non dimensional depth, located at ξl , with ξ non dimensional position.

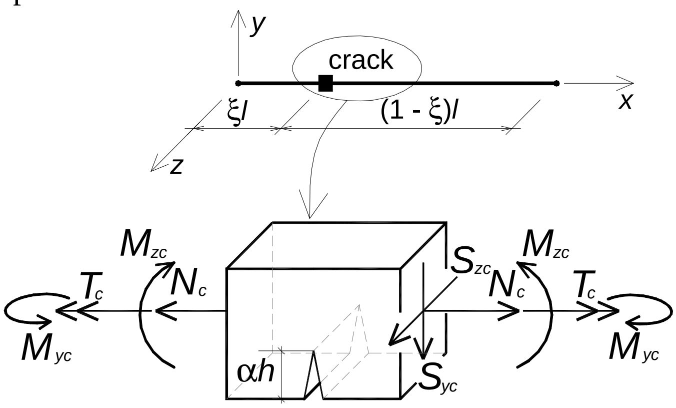

Figure 2:. Beam element with an edge crack

Due to the presence of the crack, the element compliances are expected to increase. Local compliance contributions due to the crack, depend on both the dimensionless crack depth α and location ξ. The additional compliances due to the crack are λ_N , λ_{S_y} , λ_{S_z} , λ_{M_y} , λ_{M_z} , λ_{T_x} , $\lambda_{S_zT_x}$ and λ_{NM_z} , that are the axial compliance related to axial force N, the shear compliance related to shear force S_y, the shear compliance related to shear force S_z, the bending compliance related to bending moment M_y, the bending compliance related to bending moment M_z, the torsional compliance related to torque T_x, the coupled compliance related to shear force S_z and torque T_x and the coupled compliance related to axial force N and bending moment M_z, respectively. The compliance matrix of the cracked beam is obtained by:

$$\mathbf{C}_a = \mathbf{C}_a^{in} + \mathbf{C}_a^{crack} \tag{3}$$

where $\mathbf{C}_a^{in}$ is the compliance matrix of the intact beam, whose inverse is given in Eq. 2. The local compliance contributions λ_{ij} related to the crack following the Paris's equation are given by:

$$\begin{aligned}
\lambda_{ij} &= \frac{1}{E'}\frac{\partial^2}{\partial P_i \partial P_j}\left[\int_{-B/2}^{B/2}\int_0^a\left[\left(\sum_{i=1}^{6} K_{Ii}\right)^2 + \left(\sum_{i=1}^{6} K_{IIi}\right)^2 + m\left(\sum_{i=1}^{6} K_{IIIi}\right)^2\right] da'dz\right] \\
&= \frac{1}{E'}\frac{\partial^2}{\partial P_i \partial P_j}\left[\int_{-B/2}^{B/2}\int_0^a \left(K_{I1}+K_{I2}+K_{I3}+K_{I4}+K_{I5}+K_{I6}\right)^2\right. \\
&\quad + \left(K_{II1}+K_{II2}+K_{II3}+K_{II4}+K_{II5}+K_{II6}\right)^2 \\
&\quad \left. + m\left(K_{III1}+K_{III2}+K_{III3}+K_{III4}+K_{III5}+K_{III6}\right)^2 da'dz\right]
\end{aligned} \tag{4}$$

where $E' = E$ for plane stress or $E' = E/(1-\nu^2)$ for plane strain, $m = 1+\nu$, ν is the Poisson ratio and K_{ln} are the crack stress intensity factors for the l = I, II, III modes and for n = 1,2,..,6 the load index. Note that index 1, 2, 3, 4, 5 and 6 corresponds to N, S_y, S_z, T, M_y and M_z, respectively. Many stress intensity factors are zero, in particular K_{I2}, K_{I3}, K_{I4}, K_{II1}, K_{II3}, K_{II4}, K_{II5}, K_{II6}, K_{III1}, K_{III2}, K_{III5} and K_{III6}. The remaining stress intensity factors K_{I1}, K_{I5}, K_{I6}, K_{II2}, K_{III3} and K_{III4} for the current structural configuration are given in Table 1. The compliance coefficients due to the presence of the crack are: $\lambda_N = \frac{1}{E'}\frac{2}{Bh}\int_0^a \left(F_{IN}(\alpha)\right)^2 da'$, $\lambda_{M_z} = \frac{1}{E'}\frac{12}{Bh^3}\int_0^a \left(F_{IM_z}(\alpha)\right)^2 da'$, $\lambda_{S_y} = \frac{1}{E'}\frac{2}{Bh}\int_0^a \left(F_{IIS_y}(\alpha)\right)^2 da'$, $\lambda_{M_y} = \frac{1}{E'}\frac{24}{B^3h}\int_0^a \left(F_{IM_y}(\alpha)\right)^2 da'$, $\lambda_{T_x} = \frac{m}{E'}\frac{2B}{h^3}\int_0^a \left(F_{IIIT_x}(\alpha)\right)^2 da'$.

3 PROBABILISTIC ANALYSIS

Let assume that the structural element is affected by uncertainties, which influence the compliance matrix:

$$\mathbf{C}_a = \mathbf{C}_a\left(\boldsymbol{\beta}_a\right) \tag{5}$$

where $\boldsymbol{\beta}_a$ is a vector of uncertain parameters modelled as random variables. In this paper, the uncertain parameters are the crack depth and location. The relation between nodal forces and nodal displacements is: $\mathbf{S}_a = \mathbf{K}_a\left(\boldsymbol{\beta}_a\right)\mathbf{u}_a$. Then, according to the standard matrix assembly procedure, equilibrium equations for the whole structure are obtained

$$\mathbf{K}\left(\boldsymbol{\beta}\right)\mathbf{u} = \mathbf{F} \tag{6}$$

where $\mathbf{K}(\boldsymbol{\beta})$ is the (stochastic) structure stiffness matrix, $\mathbf{u}$ is the vector of unknown nodal displacements, $\mathbf{F}$ is the vector of prescribed nodal forces and $\boldsymbol{\beta}$ is a random vector collecting variables $\boldsymbol{\beta}_a$. To characterize the structural response, nodal displacements should be evaluated as functions of the random variables $\boldsymbol{\beta}$ by solving Eq. (6). Here, the stochastic approach presented in[3-6] is followed.

Stress Intensity Factors	Geometric Functions
$K_{IN}=K_{I1}=\frac{N}{Bh^{1/2}}F_{IN}(\alpha)$ [11,12]	$F_{IN}(\alpha)=0.278\sqrt{\frac{(1+2\alpha)^3}{\alpha(1-\alpha)^3}}$
$K_{IM_z}=K_{I5}=\frac{6M_z}{Bh^{3/2}}F_{IM_z}(\alpha)$ [11,12]	$F_{IM_z}(\alpha)=\frac{0.482}{\sqrt{(1-\alpha)^3}}$
$K_{IIS_y}=K_{II2}=\frac{S_y}{Bh^{1/2}}F_{IIS_y}(\alpha)$ [11,12]	$F_{IIS_y}(\alpha)=\frac{1.2841}{\sqrt{1-\alpha}}$
$K_{IM_y}=K_{I6}=\frac{12M_y}{B^3h^{1/2}}zF_{IM_y}(\alpha)$ [10]	$F_{IM_y}(\alpha)=\sqrt{\frac{\tan(\pi\alpha/2)}{(\pi\alpha/2)}}\frac{\sqrt{\pi\alpha}}{\cos(\pi\alpha/2)}\left(0.752+2.02\alpha+0.37\left(1-\sin(\pi\alpha/2)\right)^3\right)$
$K_{IIIS_z}=K_{III3}=\frac{S_z}{Bh^{1/2}}F_{IIIS_z}(\alpha)$ [7-9]	$F_{IIIS_z}(\alpha)=\sqrt{2\tan(\pi\alpha/2)}$
$K_{IIIT_x}=\frac{T_x}{Bh^{3/2}}F_{IIIT_x}(\alpha)$ [6,11,12]	$F_{IIIT_x}(\alpha)=\frac{4.064(6k_2-1)}{(1-\alpha)^{3/2}k_2}$

Table 1 : Stress intensity factors

The basic idea is to split the element compliance matrix into a deterministic part $\mathbf{C}_a^0$ and an additional part $\mathbf{C}_a^{\beta}$ affected by uncertainty:

$$\mathbf{C}_a(\boldsymbol{\beta}_a)=\mathbf{C}_a^0+\mathbf{C}_a^{\beta}(\boldsymbol{\beta}_a) \tag{7}$$

Superscript 0 is used for deterministic quantities, while superscript β for random quantities. In the linear elastic framework, the structure is subdivided into two systems. The first system is a (reference) deterministic structure subjected to the prescribed loads $\mathbf{F}$ and ruled by the equations: $\mathbf{K}^0\mathbf{u}^0=\mathbf{F}$, $\mathbf{q}^0=\mathbf{G}^0\mathbf{u}^0$, that can be easily solved in $\mathbf{u}^0$ and $\mathbf{q}^0$ by means of standard procedures. The second system is the same deterministic structure but subjected to $\mathbf{F}^{\beta}$ instead of $\mathbf{F}$: $\mathbf{K}^0\mathbf{u}^{\beta}=\mathbf{F}^{\beta}$, $\mathbf{q}^{\beta}=\mathbf{G}^0\mathbf{u}^{\beta}+\mathbf{R}^{\beta}$. Thus, by means of the superposition principle, the expressions of $\mathbf{u}$ and $\mathbf{q}$ for the original structure take the form:

$$\mathbf{u}=\mathbf{u}^0+\mathbf{u}^{\beta}, \quad \mathbf{q}=\mathbf{q}^0+\mathbf{q}^{\beta} \tag{8}$$

The following expansion for the internal force vector $\mathbf{q}$ and for the displacement vector $\mathbf{u}$ of the original structure are obtained:

$$\mathbf{q}=\left[\sum_{j=0}^{\infty}\left(\mathbf{W}\mathbf{L}^{\beta}\right)^j\right]\mathbf{q}^0 \quad , \qquad \mathbf{u}=\mathbf{u}^0+\mathbf{U}\mathbf{L}^{\beta}\left[\sum_{j=0}^{\infty}\left(\mathbf{W}\mathbf{L}^{\beta}\right)^j\right]\mathbf{q}^0 \tag{9}$$

4 APPLICATIONS

Consider the multi-cracked frame structure shown in Fig. 3. For comparison, the results of direct Monte Carlo simulation are considered. The following data are assumed: l= 3 m, A=B×h= 0.2×0.2 m, E= 30000 N/mm², ν=0.3 and f=100 KN. The results are normalized with respect to the solution of the reference deterministic configuration. Both the location and the depth of the cracks in beams 1, 2, 3 and 4 are uncertain. All the uncertain parameters are assumed to be independent and uniformly distributed: the crack location varies in the range $\xi_{1,2,3,4} \in [0, 0.3]$ and the crack depths in the range $\alpha_{1,2,3,4} \in [0, 0.6]$.

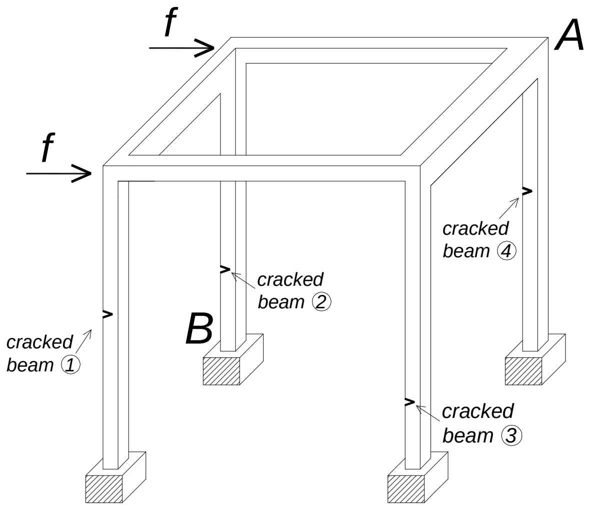

Figure 3. Three-dimensional cracked frame structure

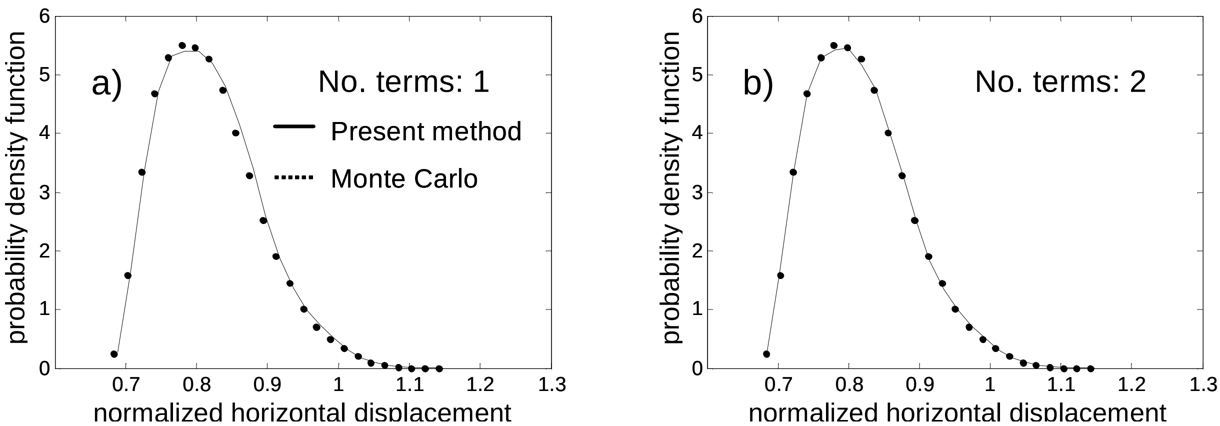

Figure 4: PDFs of the normalized horizontal displacement at node A with a) one term and b) two terms in the series.

Notice that the crack are assumed to remain open. To perform the analysis, the reference deterministic configuration should be preliminary selected according to criterion presented in[3-6]. The optimal choice results in ξ^0=0.03 and α^0=0.5067. The probability density function

for the normalized horizontal displacement u_x / u_{x0} at node A is reported in Fig. 4. From these figures it can be seen how employing just one term in the series, Eq. (9), the present method is able to reproduce the results given by the Monte Carlo simulation represented by the dotted lines. In Fig. 5, the probability density functions for the normalized bending moment m_y/m_{y0} at node B are represented. As it is expected, more terms in the series, Eq. (9), are needed to reproduce the Monte Carlo simulation results.
The comparison with classical Monte Carlo simulation evidences, for multi-cracked frame structures, the remarkable accuracy of the present approach. Another example is considered with the following data: l= 3 m, A=B×H= 0.2×0.2 m, E= 30000 N/mm², ν=0.2 and f=100 KN. Both the location and the depth of the two cracks are assumed to be independent and uniformly distributed in the range: $\alpha_{1,2} \in [0.7,1]$ and $\xi_{1,2} \in [0,0.6]$.
The results are normalized with respect to the solution of the reference deterministic configuration that is characterized by ξ^0=0.97, α^0=0.5067. The predicted distributions for the normalized axial displacement and force at node A are reported in Fig. 6 a) and b), respectively. Again, the comparison with classical Monte Carlo simulation evidences, the remarkable accuracy of the present approach.

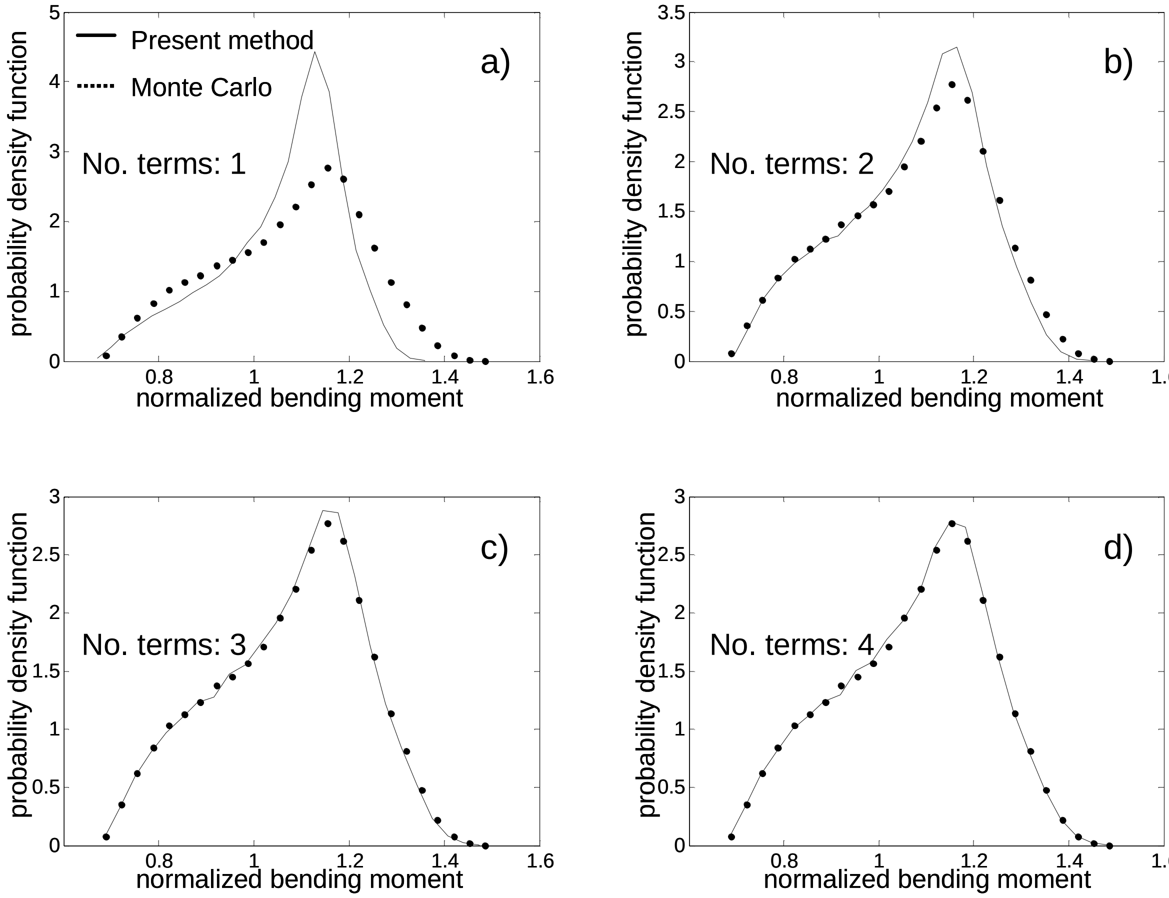

Figure 5: PDFs of the normalized bending moment at node B with a) one term, b) two, c) three and d) four terms in the series.

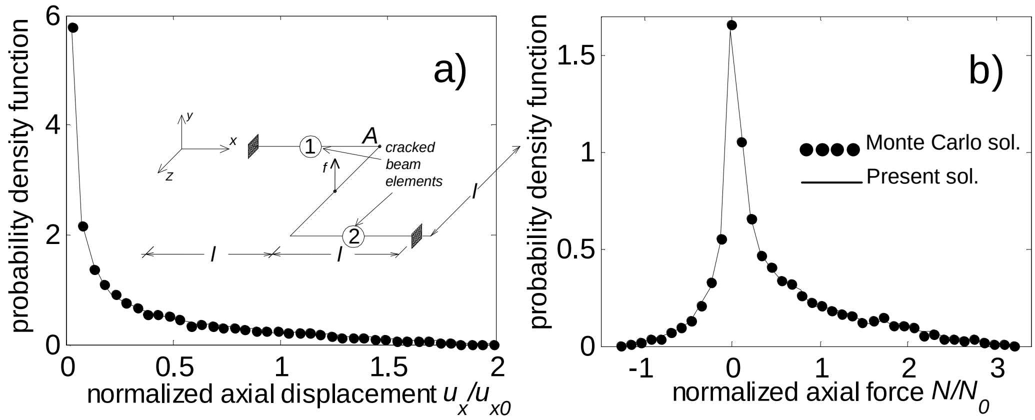

Figure 6: PDFs of the normalized axial a) displacement and b) force at node A

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