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NON-STATIONARY DYNAMIC ANALYSIS OF RANDOM STRUCTURES VIA VIRTUAL DISTORTION METHOD

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Abstract. *In this paper non-stationary dynamic analysis of linearly elastic structures involving uncertainties in material and/or in geometrical parameters has been reported in the framework of the virtual distortion method (VDM). By such a representation the statistics of the response may be obtained, explicitly, in terms of the statistics of the random parameters. The method have been extended also to the analysis of uncertain structures in presence of random external excitations providing the statistics of the response in terms of the statistics of the uncertain parameters and of the random load. The obtained stochastic moments have been contrasted with Monte-Carlo estimates used as benchmark highlighting the effects of parameter uncertainties and load fluctuations in the structural response. The use of the proposed approach to deal with Gaussian random excitation have been also reported in appendix.*

Sommario. *In questo lavoro è stata affrontata l'analisi dinamica di strutture elastiche lineari in presenza di incertezze nelle caratteristiche meccaniche e/o geometriche, mediante il metodo delle distorsioni virtuali (VDM). La presenza di fluttuazioni aleatorie nei parametri strutturali è stata quindi rappresentata da un insieme opportuno di distorsioni virtuali che dipendono dai parametri fluttuanti e dallo stato di sollecitazione e sono applicate su una struttura omogenea e deterministica. L'applicazione del metodo in campo dinamico è stata condotta mediante la trasformata di Laplace ottenendo un'espressione asintotica della risposta della struttura che contiene i parametri aleatori in forma esplicita. Di conseguenza le statistiche della risposta strutturale sono state ottenute in funzione esplicita dei momenti di ogni ordine dei parametri incerti. Sono state anche riportate alcune applicazioni numeriche relative ad applicazioni del metodo a sistemi strutturali costituiti da travi reticolari.*

1 INTRODUCTION

Random vibrations of engineering systems have been investigated since the middle of the last century in aerospace, civil and mechanical applications. The analysis, initially confined to

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Gaussian random processes to model environmental actions applied on dynamical systems, has been further extended to other kind of excitations that are better modelled as Poissonian random excitations or, more recently, as $\alpha-$ sTable Lèvy flights. In such cases the external excitations are the only source of randomness and mechanical and inertial characters of the structures were assigned up to a certain degree of accuracy. On the other hand it is nowadays widely accepted that uncertain sources in structural dynamics are related either to fluctuations of external excitations as well as to the unpredicTable deviations of material properties from their nominal values or to slightly different manufacturing procedures. Structural uncertainties, often modelled as random deviations, may significantly affect the dynamic response of the considered system leading to consider probabilistic studies of random structures since the seventies.

Probabilistic structural analysis in presence of random parameters is usually provided in terms of first and second order statistics of the displacement field of the structure once the probabilistic structure of the random parameters has been specified. In such a context several contributions may be found in scientific literatures initially focusing on Monte-Carlo simulation methods[1]. Such a rather general tool to handle random systems proved soon to be unpractical for large engineering systems with may degree of freedom since it requires thousands of time consuming analysis to yield accurate description of the statistics of the structural response. Such a consideration suggested the introduction of other, more efficient tools for the analysis of randomly fluctuating structures. Those approaches may be framed in the context of the finite element method for stochastic structure since they involve random stiffness matrices of the structure that cannot be inverted in closed, explicit form of the random parameters. This unavoidable aspect in the analysis of random structures led to introduce first and second order perturbation methods, dubbed respectively FORM and SORM[2-5] that involve a Taylor series expansion of the stiffness matrix in terms of the uncertain parameters truncated, respectively, to first and second order terms. As an alternative the inverse of the random stiffness matrix of the structure has been expanded in Neumann series[6,7] involving matrix powers of the random stiffness matrix that lead various authors to use it in symbiosis with Monte-Carlo analysis. Some attempts to introduce exact expressions for the statistics of the displacement field of random structures has been also furnished in the last decade in presence of statically determinate structures[8-10] . More recently a different method based on the eigenproperties of the random stiffness element matrices of the structure has been introduced both in static[11] and dynamic setting[12,13] . Random uncertainties in structural parameters and random dynamic external excitations have been investigated in the context of improved perturbation methods as well as in presence of random damping characteristics. In more recent studies a method based on stochastic differential calculus to handle random vibrations of uncertain structures have been also proposed[14].

In recent years analysis of uncertain trusses has been framed in the context of the virtual distortion method (VDM) that is a method to study non-homogeneous solids, i.e. in presence of inclusions, originally proposed in the context of solid mechanics[15].Such a method aims to represent the effects of solid-body inclusions as a convenient set of superimposed strain field depending of the strain field and of the inclusion characteristics. The method, applied to truss and frame-like structures is specialized considering that the elongation of a bar with uncertain parameters may be evaluated considering the bar without parameter variations with a

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convenient superimposed virtual distortion. Such distortions, elongations for a truss-like structure, are parameter and axial stress dependent and such an analysis has been widely used in the analysis of random structures in static setting yielding exact closed-form expressions of the statistics of the displacement field for statically determinate structures. Analysis of statically indeterminate structures leads to an asymptotic expansion of the statistics of the response in terms of the known statistics of the uncertain parameters[16]. The method has been successfully applied to elastic uncertain trusses, both under deterministic and random static loads[17] and to the cases of dynamic stationary vibrations under deterministic load[18-21].
The aim of the paper is to extend the VDM to the dynamic analyses of randomly fluctuating structures in presence of uncertain parameters and non-stationary excitations. The analysis has been carried out in Laplace domain taking full advantages of convolution theorems and evaluating the displacements of the structure by a matrix power series. Every-order statistics of the structural response may be obtained with the proposed approach by means of Kronecker algebra and they involve, in explicit form, the statistical moments of the random structural parameters.
The paper reports the general preliminaries to the VDM in sec.2. Sec.3 has been devoted to the analysis of a random structure providing the necessary formulation in terms of statistical moments. The influence of the randomness of the structural parameters on the quality of the response has been also shown for a nine-degree of freedom structure reporting the first and second-order moments of the displacement vector. The response evaluated with the proposed method in the present formulation have been contrasted with the statistics obtained via Monte-Carlo simulations used as benchmark solution; some conclusions has been reported in sec.5. In appendix A1 some details about Kronecker algebra have been reported and a possible extension of the proposed method in presence of Gaussian random loads has been reported in appendix A2.

2 THE VIRTUAL DISTORTION METHOD: THEORETICAL BACKGROUND

In this section VDM is introduced to deal with forced vibration of structural systems experiencing variation in characteristic parameters like elastic modulus and/or cross-sections dimensions. Fundamentals of VDM may be expressed resorting to a linear elastic system like the schematic truss depicted in Fig.1. In particular this truss is composed by N nodes in which the masses and the external loads are concentrated and connected by N massless bars (N = 13 in Fig.1)
The dynamic equilibrium of the structure is ruled by the system of coupled differential equations:

$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}}(B,t)+\mathbf{D}\dot{\mathbf{u}}(B,t)+\mathbf{K}(B)\mathbf{u}(B,t)=\mathbf{f}(t) \\ \mathbf{u}(B,0)=\mathbf{u}_0;\dot{\mathbf{u}}(B,0)=\dot{\mathbf{u}}_0 \end{cases} \tag{1}$$

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where we denoted $\mathbf{u}(B,t)$ the ($N \times 1$) displacement vector depending on time t and parameters variations indicated by B, $\mathbf{f}$(t) the ($N \times 1$) vector gathering the external dynamic loads, $\mathbf{M}$, $\mathbf{D}$, $\mathbf{K}$ are the mass, dissipation and stiffness matrix of the truss, respectively. In particular the latter one is defined as $\mathbf{K}(B)=\mathbf{C}^T\mathbf{E}(B)\mathbf{C}$ being $\mathbf{C}^{\mathrm{T}}$ the $(N \times N)$ equilibrium matrix of the truss and $\mathbf{E}(B)$ is the $(N \times N)$ diagonal constitutive matrix of the truss with elements $E_{jj}=\bar{E}_j\bar{A}_j(1+B_j)/L_j$ with E_j modulus of elasticity, A_j the cross-sectional area and L_j the length of the j^{th} member of the truss, respectively. Of primary importance is the role of B_j representing the parameter variation, that may be modelled by a random variable with prescribed probability density function (pdf).

The analysis of such a structure will be completed once the response is evaluated in terms of displacements $\mathbf{u}(t,B)$, elongations $\mathbf{e}(t,B)$ and axial stresses $\mathbf{q}(t,B)$ that are of course parameter dependent.

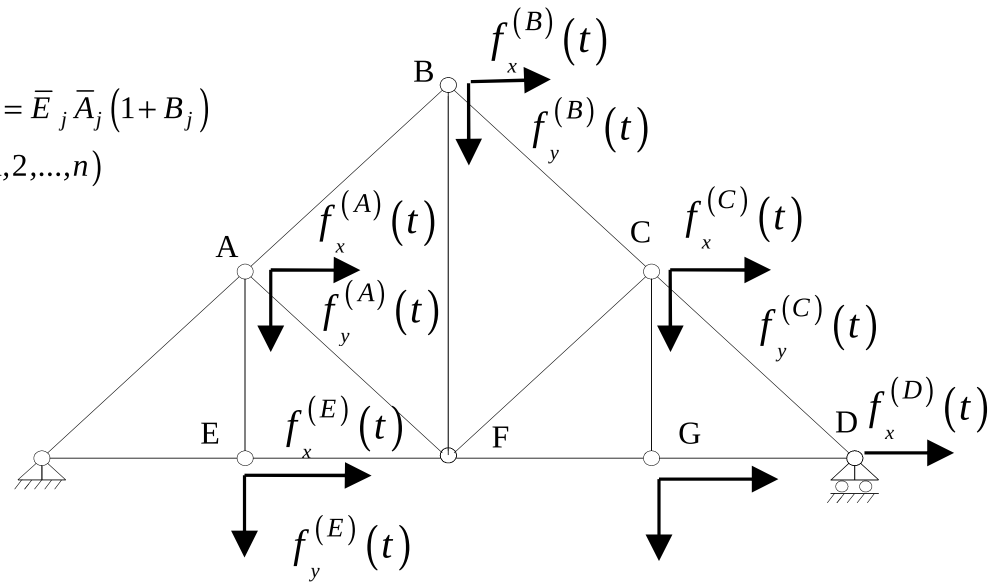
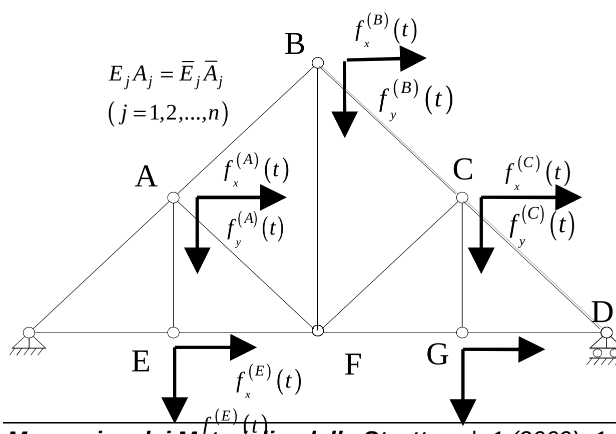


Figure 1a: Sample of multi-degree of freedom truss with randomly varying parameters



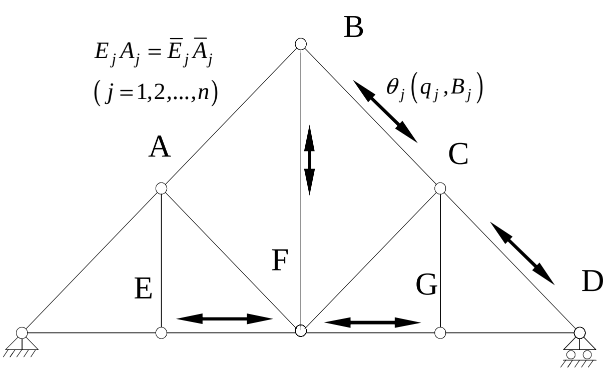


Fig.1b: Principal Structure   Fig.1c: Auxiliary Structure

Performing analysis in the context of VDM [1] , in linear setting so that the superposition principle holds, we can solve this problem by two analysis of the same structure, but without parameters variation, under two different agencies. Therefore we have to consider the structure with no parameters variation as *reference structure,* that differs from the original one only for the stiffness matrix $\mathbf{K}=\mathbf{C}^T\bar{\mathbf{E}}\mathbf{C}$ that doesn't depend on B_j being $\bar{\mathbf{E}}$ populated by diagonal elements $\bar{E}_{jj}=\bar{E}_j\bar{A}_j/L_j$ that is, all the parameter variations are setting zeros.
Firstly we consider this reference structure loaded by the external agencies Fig.(1 b) of the original system that will be referred as *principal structure* whose governing equation is in the form:

$$\begin{cases}\mathbf{M}\ddot{\mathbf{u}}_{\mathrm{p}}(\mathrm{t})+\mathbf{D}\dot{\mathbf{u}}_{\mathrm{p}}(\mathrm{t})+\mathbf{K}\mathbf{u}_{\mathrm{p}}(\mathrm{t})=\mathbf{f}(\mathrm{t})\\ \dot{\mathbf{u}}_{\mathrm{p}}(0)=\dot{\bar{\mathbf{u}}};\mathbf{u}_{\mathrm{p}}(0)=\bar{\mathbf{u}}\end{cases} \tag{2}$$

Secondly we study the same structure, labelled as *auxiliary structure* Fig.(1 c) under a proper load that takes into account parameters variation:

$$\begin{cases}\mathbf{M}\ddot{\mathbf{u}}_a(t)+\mathbf{D}\dot{\mathbf{u}}_a(t)+\mathbf{K}\mathbf{u}_a(t)=\mathbf{p}(t,B)\\ \dot{\mathbf{u}}_a(0)=0;\mathbf{u}_a(0)=0\end{cases} \tag{3}$$

For the definition of vector $\mathbf{p}(t,B)$ it is worth stressing that in static setting the same problem has been solved by VDM [16] applying superimposed strains of the form:

$$\boldsymbol{\vartheta}(B)=-\bar{\mathbf{E}}^{-1}\mathbf{L}(B)\mathbf{q}(t,B) \tag{4}$$

on the *reference structure,* with the $(N\times N)$ parameter dependent diagonal matrix $\mathbf{L}(B)$ listing elements of type $L_{jj}=B_j/(1+B_j)$ in correspondence of members affected by random parameter variation B_j. Close observation of eq.(4) shows that superimposed strain vector $\boldsymbol{\vartheta}(B)$ depends on still unknown real stress $\mathbf{q}(t,B)$ of the original structure.

Extension to dynamic setting of the VDM leads to set the load vector $\mathbf{p}(t,B)$ in eq (3) as:

$$\mathbf{p}(t,B)=-\mathbf{C}^T\mathbf{L}(B)\mathbf{q}(t,B) \tag{5}$$

According to the superposition principle, internal axial stress vector $\mathbf{q}(t,B)$ the displacement vector $\mathbf{u}(t,B)$ and the strains $\mathbf{e}(t,B)$ are evaluated adding the contributions of the principal and the auxiliary structure in the form as:

$$\mathbf{q}(t,B)=\mathbf{q}_p(t)+\mathbf{q}_a(t,B) \tag{6 a}$$

$$\mathbf{u}(t,B)=\mathbf{u}_p(t)+\mathbf{u}_a(t,B) \tag{6 b}$$

$$\mathbf{e}(t,B)=\mathbf{e}_p(t)+\mathbf{e}_a(t,B) \tag{6 c}$$

2.1 PRINCIPAL STRUCTURE RESPONSE

The evaluation of the principal structure response, namely vectors $\mathbf{u}_p(t)$, $\mathbf{q}_p(t)$ and $\mathbf{e}_p(t)$ requires solution of the coupled differential equations system in eq.(2). The problem may be afforded by classical modal analysis using the coordinate transformation $\mathbf{u}_p(t)=\boldsymbol{\Phi}\,\mathbf{y}_p(t)$, where matrix $\boldsymbol{\Phi}$ is the deterministic modal matrix collecting the eigenvectors of the matrix $\mathbf{K}^{-1}\mathbf{M}$.

Therefore introducing the above transformation into eq.(2) and multiplying both sides of the resulting equation by matrix $\boldsymbol{\Phi}^T$ yielding:

$$\begin{cases}\ddot{\mathbf{y}}_p(t)+\boldsymbol{\Lambda}\dot{\mathbf{y}}_p(t)+\boldsymbol{\Omega}^2\mathbf{y}_p(t)=\boldsymbol{\Phi}^T\mathbf{f}(t)=\mathbf{g}(t)\\ \mathbf{y}_p(0)=\boldsymbol{\Phi}^T\mathbf{u}_p(0);\dot{\mathbf{y}}_p(0)=\boldsymbol{\Phi}^T\dot{\mathbf{u}}_p(0)\end{cases} \tag{7}$$

where $\boldsymbol{\Lambda}$, $\boldsymbol{\Omega}^2$ are diagonal matrices listing, respectively:

$$\boldsymbol{\Lambda}=\ulcorner 2\zeta_1\omega_1\ 2\zeta_2\omega_2\ \ldots\ \ldots\ 2\zeta_N\omega_N \lrcorner \tag{8 a}$$

$$\boldsymbol{\Omega}^2=\ulcorner \omega_1^2\ \omega_2^2\ \ldots\ \ldots\ \omega_N^2 \lrcorner \tag{8 b}$$

being ζ_j the *j*-th dissipation coefficient and ω_j^2 the *j*-th eigenvalue of the matrix $\mathbf{K}^{-1}\mathbf{M}$.

Moreover, assuming that the system is at rest in its initial state, that is $\mathbf{u}_p(0)=\mathbf{u}_0=\mathbf{0}$; $\dot{\mathbf{u}}_p(0)=\dot{\mathbf{u}}_0=\mathbf{0}$ then the displacement vector response of the principal structure is obtained via Duhamel integral as:

$$\mathbf{u}_p(t)=\boldsymbol{\Phi}\,\mathbf{y}_p(t)=\boldsymbol{\Phi}\int_0^t\mathbf{H}(t-\tau)\mathbf{g}(\tau)\,\mathrm{d}\tau=\boldsymbol{\Phi}\mathbf{H}(t)*\mathbf{g}(t) \tag{9}$$

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where $[\square]*[\square]$ is the faltung product and $\mathbf{H}(t) = \lceil h_{11}(t) \quad h_{22}(t) \quad \ldots \quad h_{NN}(t) \rfloor$ is a diagonal matrix gathering the impulse response functions $h_{jj}(t)$ of the modal coordinates $y_{p_j}(t)$:

$$h_{jj}(t) = \exp(-\zeta_j \omega_j t) \frac{\sin(\bar{\omega}_j t)}{\bar{\omega}_j} \tag{10}$$

being the damped natural frequency $\bar{\omega}_j = \omega_j \sqrt{1-\zeta_j^2}$.

To avoid the convolution integral in eq.(9) Laplace transform is used in the well-known direct and inverse form as:

$$\hat{w}(s) = \int_0^\infty e^{-st} w(t)\ dt = \wp[w(t)] \tag{11 a}$$

$$w(t) = \int_0^\infty e^{st} \hat{w}(s)\ ds = \wp^{-1}[\hat{w}(s)] \tag{11 b}$$

being s the complex Laplace parameter. Introducing Laplace transform to both sides of eq.(9) and taking full advantage of convolution theorem [1] , the nodal displacement vector may be reported in the form:

$$\hat{\mathbf{u}}_p(s) = \mathbf{\Phi}\hat{\mathbf{y}}_p(s) = \mathbf{\Phi}\hat{\mathbf{H}}(s)\hat{\mathbf{g}}(s) \tag{12}$$

where matrix $\hat{\mathbf{H}}(s) = \wp[\mathbf{H}(t)]$ collects the Laplace transforms of the impulse response function $h_{jj}(t)$ expressed as:

$$\hat{h}_{jj}(s) = \wp[h_{jj}(t)] = \frac{1}{(\omega_j^2 + s^2 + 2\zeta_j \omega_j s)} \tag{13}$$

2.2 AUXILIARY STRUCTURE RESPONSE

The evaluation of the auxiliary structure response, namely vectors $\mathbf{u}_a(t,B)$, $\mathbf{q}_a(t,B)$ and $\mathbf{e}_a(t,B)$ is not trivial, since the elements of vector $\mathbf{p}(t,B)$ depend on the axial stress $\mathbf{q}(t,B)$ which is still unknown, similar to the analysis of statically indeterminate trusses (Di Paola, 2004). In the following dynamic analysis vector $\mathbf{p}(t,B)$ that represents the inertial, time-dependent, virtual distortions depends on the axial stress and may be conveniently expressed in terms of nodal displacements as:

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$$\mathbf{p}(t,B) = -\mathbf{C}^T\mathbf{B}\bar{\mathbf{E}}\mathbf{C}\mathbf{u}(t,B) = -\mathbf{C}^T\mathbf{B}\bar{\mathbf{E}}\mathbf{C}\left[\mathbf{u}_p(t) + \mathbf{u}_a(t,B)\right] \tag{14}$$

with the parameter-dependent $(N \times N)$ diagonal matrix $\mathbf{B} = \lceil B_1 \quad B_2 \quad \dots \quad B_N \rfloor$, collecting random fluctuations of structural parameters, the governing equation of the auxiliary structure is rewritten in the form:

$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}}_a(t) + \mathbf{D}\dot{\mathbf{u}}_a(t) + \mathbf{K}\mathbf{u}_a(t) = -\mathbf{C}^T\mathbf{B}\bar{\mathbf{E}}\mathbf{C}\left[\mathbf{u}_p(t) + \mathbf{u}_a(t)\right] \\ \dot{\mathbf{u}}_a(0) = 0; \mathbf{u}_a(0) = 0 \end{cases} \tag{15 a, b}$$

Dynamic analysis of the auxiliary structure will be performed introducing the modal transformation of the displacement vector $\mathbf{u}_a(t,B)$ with the modal matrix of the reference structure, namely $\mathbf{\Phi}$ as:

$$\mathbf{u}_a(t,B) = \mathbf{\Phi}\,\mathbf{y}_a(t,B) \tag{16}$$

that substituted into eq.(15 a, b) yields, after some straightforward algebra, the system of differential equations in the modal space as:

$$\begin{cases} \ddot{\mathbf{y}}_a(t,B) + \mathbf{\Lambda}\dot{\mathbf{y}}_a(t,B) + \mathbf{\Omega}^2\mathbf{y}_a(t,B) = \mathbf{R}(B)\left(\mathbf{y}_p(t) + \mathbf{y}_a(t,B)\right) \\ \mathbf{y}_a(0,B) = \mathbf{0}; \dot{\mathbf{y}}_a(0,B) = \mathbf{0} \end{cases} \tag{17}$$

with the parameter-dependent matrix $\mathbf{R}(B)$ represented as:

$$\mathbf{R}(B) = \mathbf{\Gamma}^T\mathbf{B}\bar{\mathbf{E}}\mathbf{\Gamma} \quad ; \quad \mathbf{\Gamma} = \mathbf{C}\mathbf{\Phi} \tag{18 a, b}$$

Eq.(17) is a system of second-order coupled differential equations since load vector at the right-hand side involves modal coordinates in every equation. Integral representation of the solution of eq.(17) may be obtained via Duhamel integral resulting into a Volterra integral equations system of second kind:

$$\mathbf{y}_a(t,B) = -\int_0^t \mathbf{H}(t-\tau)\mathbf{R}(B)\mathbf{y}_p(\tau)\,\mathrm{d}\tau - \int_0^t \mathbf{H}(t-\tau)\mathbf{R}(B)\mathbf{y}_a(\tau)\,\mathrm{d}\tau \tag{19}$$

Laplace transform of both sides of eq.(19) yields, using convolution theorem, to the system of coupled *N* algebraic equations:

$$\hat{\mathbf{y}}_a(s,B) = -\hat{\mathbf{H}}(s)\mathbf{R}(B)\hat{\mathbf{y}}_p(s) - \hat{\mathbf{H}}(s)\mathbf{R}(B)\hat{\mathbf{y}}_a(s,B) \tag{20}$$

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that may be solved, by means of successive approximations assuming initially that no the structural response vector $\hat{\mathbf{y}}_a^{(1)}$ is provided only by the first term in eq.(20) that is known. As we obtained $\hat{\mathbf{y}}_a^{(1)}$ we may evaluate a better approximation using it as a known corrective term provided by the second contribution in eq.(20) yielding the following iteration scheme as from Picard' method:

$$\begin{aligned}
&\hat{\mathbf{y}}_a^{(1)}(s,B) = -\hat{\mathbf{H}}(s)\mathbf{R}(B)\hat{\mathbf{y}}_p(s) = -\hat{\mathbf{H}}(s)\mathbf{R}(B)\hat{\mathbf{H}}(s)\hat{\mathbf{g}}(s)\\
&\hat{\mathbf{y}}_a^{(2)}(s,B) = -\hat{\mathbf{H}}(s)\mathbf{R}(B)\hat{\mathbf{H}}(s)\hat{\mathbf{g}}(s) - \hat{\mathbf{H}}(s)\mathbf{R}(B)\hat{\mathbf{y}}_a^{(1)}(s,B)\\
&\ldots\\
&\hat{\mathbf{y}}_a^{(r)}(s,B) = -\hat{\mathbf{H}}(s)\mathbf{R}(B)\hat{\mathbf{H}}(s)\hat{\mathbf{g}}(s) - \hat{\mathbf{H}}(s)\mathbf{R}(B)\hat{\mathbf{y}}_a^{(r-1)}(s,B)\\
&\ldots
\end{aligned} \tag{21}$$

that may be cast in compact, form as:

$$\hat{\mathbf{y}}_a(s,B) = \sum_{j=1}^{\infty}\left(-\hat{\mathbf{H}}(s)\mathbf{R}(B)\right)^j \hat{\mathbf{H}}(s)\hat{\mathbf{g}}(s) \tag{22}$$

Once the dynamic modal displacement vector for the auxiliary structure has been obtained by eq.(22), then the nodal displacement vector of the auxiliary structure, $\hat{\mathbf{u}}_a(s,B)$, is furnished by eq.(16).

2.3 STRUCTURAL RESPONSE

The complete dynamic response of the structure, in Laplace domain, is obtained via eq.(6 a) substituting the expressions for the modal coordinates obtained in eqs.(12 ,22) yielding:

$$\hat{\mathbf{u}}(s,B) = \hat{\mathbf{u}}_p(s) + \hat{\mathbf{u}}_a(s,B) = \mathbf{\Phi}\,\hat{\mathbf{H}}(s)^{1/2}\sum_{j=0}^{\infty}\left(-\mathbf{A}^T(s)\mathbf{B}\mathbf{A}(s)\right)^j \hat{\mathbf{H}}(s)^{1/2}\,\mathbf{\Phi}^T\hat{\mathbf{f}}(s) \tag{23}$$

where $\mathbf{A}(s) = \overline{\mathbf{E}}^{1/2}\mathbf{\Gamma}\left(\hat{\mathbf{H}}(s)\right)^{1/2}$. It is worth noticing that expansion in eq.(23) is always convergent provided that the maximum value of the fluctuations of the structural parameters satisfies the condition $max\left|B_j\right| \le 1$; j=1,2,…,N, since the maximum value of the spectral radius, represented y the maximum value of the modulus of the eigenvalues of the matrix

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$\mathbf{A}^T(s)\mathbf{B}\mathbf{A}(s)$ and dubbed $\lambda(s,B)$ is obtained for $s=0$ and takes value $\lambda(0,B)=\max_{j=1,2,...,N}\left|B_j\right|$.

Under the latter conditions the expansion in eq.(23) quickly yields the state variable vector $\hat{\mathbf{u}}(s,B)$ and in applications an approximate form of eq.(23) may be used considering N_{max} as the maximum number of iterations.

At this point some remarks shall be stressed: *i*) Analysis conducted with the aid of Laplace transform is not affected by drawbacks existent in a previously used Fourier transform (Di Paola *et al*., 2004) approach since the spectral radius of the general term in series expansion in eq.(23) does not depend on external agencies; *ii*) eq.(23) is still useless for probabilistic analysis since no explicit dependence on structural parameters has been provided.

In the next sections this major drawback will be overcome introducing some Kronecker algebra to inflate structural parameter space obtaining explicit expressions for the statistics of the structural response.

3 PROBABILISTIC ANALISYS OF RANDOM TRUSSES

Let us assume that random parameters $B_j \ (j=1,2,...,N)$ belong to a symmetric, closed, interval $B_j \in \left[-\bar{b}_j,\bar{b}_j\right]$ and they are collected in a *N*-vector $\mathbf{b}^T=\left[B_1 \quad B_2 \quad ... \quad B_N\right]$ with prescribed joint probability density function (pdf) denoted $p_{\mathbf{b}}(b_1,b_2,...,b_N)$. Probabilistic analysis of random structures may be conducted once we get rid of the matrix products and powers involving random parameter matrix in eq.(23). Such a consideration may be provided observing that each term in the matrix $\mathbf{A}^T(s)\mathbf{B}\mathbf{A}(s)$ is a linear combination of the uncertain parameters B_j so that the following equality holds true:

$$\sum_{k=1}^{N}\frac{\partial\left(\mathbf{A}^T(s)\mathbf{B}\mathbf{A}(s)\right)}{\partial B_k}B_k=\sum_{k=1}^{N}\mathbf{A}_k(s)B_k=\mathbf{A}^T(s)\mathbf{B}\mathbf{A}(s) \tag{24}$$

where matrix $\mathbf{A}_k(s)$ reads:

$$\mathbf{A}_k(s)=\begin{bmatrix} a_{1k}(s)a_{k1}(s) & a_{1k}(s)a_{k2}(s) & ... & a_{1k}(s)a_{kN}(s) \\ a_{2k}(s)a_{k1}(s) & a_{2k}(s)a_{k2}(s) & ... & a_{2k}(s)a_{kN}(s) \\ ... & ... & ... & ... \\ a_{Nk}(s)a_{k1}(s) & a_{Nk}(s)a_{k2}(s) & ... & a_{Nk}(s)a_{kN}(s) \end{bmatrix} \tag{25}$$

and $a_{kj}(s) \ (k,j=1,2,...,N)$ is the jk-element of the matrix $\mathbf{A}(s)$. Introducing Kronecker algebra eq.(24) may be used as parent expression for the j-th power in series expansion reported in eq.(24) observing that:

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$$\mathbf{A}^T(s)\mathbf{B}\mathbf{A}(s)=\sum_{k=1}^{N}\mathbf{A}_k(s)B_k=\mathbf{A}_e(s)\mathbf{b}\otimes\mathbf{I} \tag{26}$$

where matrix $\mathbf{I}$ is the $(N\times N)$ identity matrix and the $(N\times N^2)$ matrix $\mathbf{A}_e(s)$, is the block matrix reading:

$$\mathbf{A}_e(s)=\left[\mathbf{A}_1(s)\quad \mathbf{A}_2(s)\quad \ldots\quad \mathbf{A}_N(s)\right] \tag{27}$$

and operator $[\square]\otimes[\square]$ is the Kronecker product. Some details about Kronecker algebra have been reported in the appendix A1. Following expansion in eq.(26) the j-th power in eq.(23) may be rewritten as:

$$\left(\mathbf{A}^T(s)\mathbf{B}\mathbf{A}(s)\right)^j=\mathbf{A}_e^{(j)}(s)\mathbf{b}^{[j]}\otimes\mathbf{I} \tag{28}$$

where $[\square]^{[j]}$ denotes Kronecker power of order *j* and matrix $\mathbf{A}_e^{(j)}(s)$ is obtained via iterative formula as:

$$\begin{aligned}&\mathbf{A}_e^{(j)}(s)=\mathbf{A}_e^{(1)}(s)\left(\mathbf{I}\otimes\mathbf{A}_e^{(j-1)}(s)\right)\\&\mathbf{A}_e^{(j-1)}(s)=\mathbf{A}_e^{(1)}(s)\left(\mathbf{I}\otimes\mathbf{A}_e^{(j-2)}(s)\right)\\&\ldots\\&\mathbf{A}_e^{(1)}(s)=\mathbf{A}_e(s)\end{aligned} \tag{29}$$

Substitution of eq.(28) into eq.(23) yields the nodal displacement vector in the form:

$$\hat{\mathbf{u}}(s)=\hat{\boldsymbol{\Phi}}(s)\left(\sum_{j=0}^{\infty}(-1)^j\,\mathbf{A}_e^{(j)}(s)\mathbf{b}^{[j]}\otimes\mathbf{I}\right)\hat{\boldsymbol{\Phi}}(s)^T\,\hat{\mathbf{f}}(s) \tag{30}$$

where matrix $\hat{\boldsymbol{\Phi}}(s)=\boldsymbol{\Phi}\hat{\mathbf{H}}(s)^{1/2}$. Eq.(30) is an explicit formula in terms of parameter fluctuations and it may be used to evaluate the statistics of the nodal displacement vector $\mathbf{u}(t)$. To this aim let us evaluate the inverse Laplace transform to eq.(30) yielding:

$$\mathbf{u}(t)=\wp^{-1}\left[\hat{\boldsymbol{\Phi}}(s)\left(\sum_{j=0}^{\infty}(-1)^j\,\mathbf{A}_e^{(j)}(s)\mathbf{b}^{[j]}\otimes\mathbf{I}\right)\hat{\boldsymbol{\Phi}}(s)^T\,\hat{\mathbf{f}}(s)\right] \tag{31}$$

and performing mathematical expectation the mean displacement vector reads:

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$$\boldsymbol{\mu}_{\mathbf{u}}(t) = E\left[\mathbf{u}(t)\right] = E\left[\wp^{-1}\left[\hat{\boldsymbol{\Phi}}(s)\left(\sum_{j=0}^{\infty}(-1)^{j}\,\mathbf{A}_{e}^{(j)}(s)\,\mathbf{b}^{[j]}\otimes\mathbf{I}\right)\hat{\boldsymbol{\Phi}}(s)^{T}\,\hat{\mathbf{f}}(s)\right]\right] =$$
$$= \wp^{-1}\left[\hat{\boldsymbol{\Phi}}(s)\left(\sum_{j=0}^{\infty}(-1)^{j}\,\mathbf{A}_{e}^{(j)}(s)\,\mathbf{m}_{j}(\mathbf{b})\otimes\mathbf{I}\right)\hat{\boldsymbol{\Phi}}(s)^{T}\,\hat{\mathbf{f}}(s)\right] \quad (32)$$

where mathematical expectation $E\left[\mathbf{b}^{[j]}\right] = \mathbf{m}_{j}(\mathbf{b})$ is the vector of stochastic moments of order j of the random parameter vector that is known once the pdf $p_{\mathbf{b}}(b_1, b_2, ..., b_N)$ has been prescribed.

Second-order moments of the displacement vector defined as $E\left[\mathbf{u}(t_1)\otimes\mathbf{u}(t_2)\right]$ may be provided by similar operations, yielding, for the moment vector:

$$E\left[\mathbf{u}(t_1)\otimes\mathbf{u}(t_2)\right] =$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\wp^{-1}\left[(-1)^{j}\,\hat{\boldsymbol{\Phi}}(s_1)\,\mathbf{A}_{e}^{(j)}(s_1)\,\bar{\mathbf{m}}_{j}\hat{\boldsymbol{\Phi}}(s_1)^{T}\,\hat{\mathbf{f}}(s_1)\right]\otimes\wp^{-1}\left[(-1)^{k}\,\hat{\boldsymbol{\Phi}}(s_2)\,\mathbf{A}_{e}^{(k)}(s_2)\,\bar{\mathbf{m}}_{k}\hat{\boldsymbol{\Phi}}(s_2)^{T}\,\hat{\mathbf{f}}(s_2)\right] \quad (33)$$

where we denoted the $\bar{\mathbf{m}}_{j}$ and $\bar{\mathbf{m}}_{k}$, respectively, $\left(N^{j}\times N\right)$ and $\left(N^{k}\times N\right)$ matrices obtained as matrix blocks:

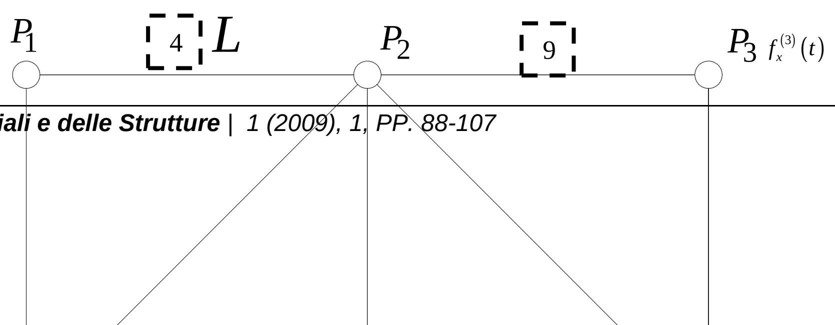
$$\mathbf{m}_{j+k}(\mathbf{b})\otimes\mathbf{I} = \begin{bmatrix}\bar{\mathbf{m}}_{j}\\ \bar{\mathbf{m}}_{k}\end{bmatrix} \quad (34)$$

Multiple-time expectation may be provided with similar formulas and they have not been reported for brevity.

The case of non-stationary random excitation under univariate-Gaussian random load has been reported in appendix B since it can be derived straightforwardly within the context of proposed method.

4 NUMERICAL APPLICATIONS

In this section the proposed method to deal with random vibrations of random structures will be used to represent the random response of the 9-degree of freedom uncertain truss reported in Fig.(3).



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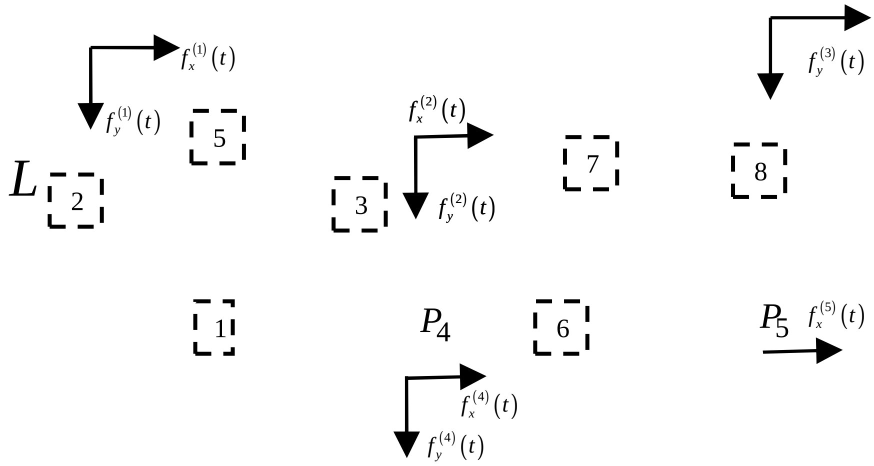

Bar Label	L_i (cm)	A_i (cm^2)	E_i (Kg/cm^2)	α_i (rad)	$\bar{b}_i$ (%)
1	250	4	2.1×10^6	0	10%
2	250	4	2.1×10^6	$\pi/2$	20%
3	250	4	2.1×10^6	$\pi/2$	30%
4	250	4	2.1×10^6	π	10%
5	$250\sqrt{2}$	8	2.1×10^6	$\pi/4$	20%
6	250	4	2.1×10^6	0	30%
7	$250\sqrt{2}$	8	2.1×10^6	$3/4\pi$	10%
8	250	4	2.1×10^6	$\pi/2$	20%
9	250	4	2.1×10^6	π	30%

Tab.1: Bar properties of the nine degree of freedom truss

The geometric and elastic characteristics of the truss members have been reported in Tab.1. Each truss member is affected by a structural uncertainty so that the axial stiffness of the bar is modelled as $\bar{E}_j\bar{A}_j\left(1+B_j\right)/L_j$ with B_j $\left(j=1,2,...,9\right)$ mutually independent random variables $E\left[B_jB_k\right]=\delta_{jk}\sigma_j^2$ $\left(j,k=1,2,...,9\right)$ where σ_k^2 is the standard deviation of the the *k*-th random variable and δ_{jk} is the well-known Kronecker delta. Random variables B_j $\left(j=1,2,...,9\right)$ are characterized by means of a prescribed, uniform, probability density function as $p_{B_j}\left(b_j\right)=1/2\,\bar{b}_j$ $\quad b_j\in\left[-\bar{b}_j,\bar{b}_j\right]$.

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Node	$f_x\ (Kg_f)$	$f_y\ (Kg_f)$	$M\ (Kg_m)$
P_1	0	0	600
P_2	0	1000	500
P_3	0	0	600
P_4	0	0	500
P_5	0	0	600

Tab.2: Nodal Loads and Nodal Masses

The values of maximum amplitude of parameter fluctuations as well as the values of the coefficient of variations ($\mathrm{cov}_k = \sigma_k = \bar{b}_k / 3$) of the random fluctuations have been reported in the last two columns of Tab.(1) as well.

The random load applied to the truss nodes is represented as an 1-variate random vector process as $\mathbf{f}(t)^T = f(t)\left[\mathbf{f}_1 \quad \mathbf{f}_2 \quad \mathbf{f}_3 \quad \mathbf{f}_4 \quad f_x^{(5)}\right]^T$ where the component vectors $\mathbf{f}_j = \left[f_x^{(j)} \quad f_y^{(j)}\right]$ $(j = 1,...,4)$ represents the loads applied to the nodes of the truss and they have been specified in Tab.(2) that reports also the nodal masses used for dynamic analysis.

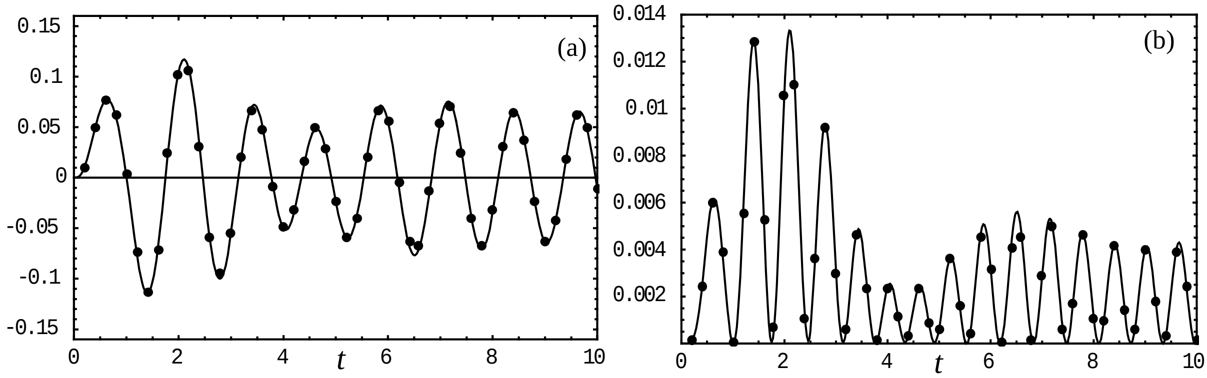

Figure 4 a,b: Mean and Second-Order moment of vertical displacement $U_2(t)$; Continuous line proposed method, Dots Monte-Carlo Simulation

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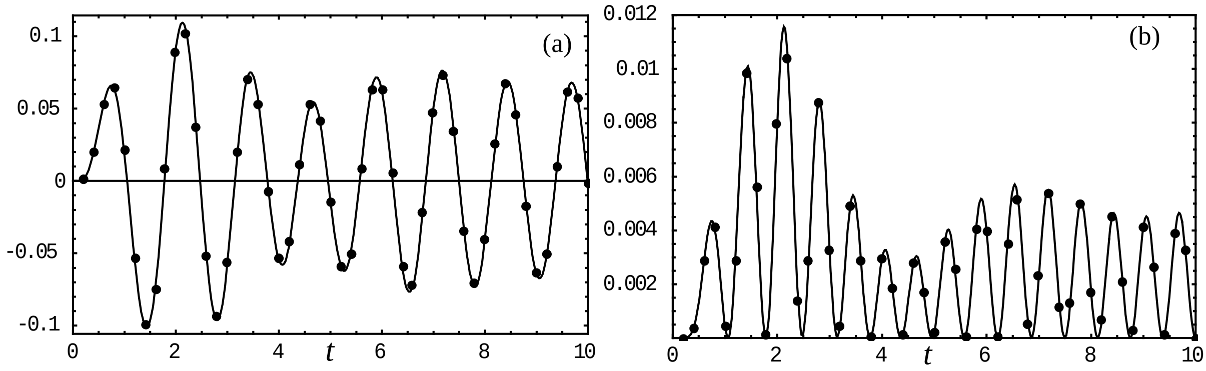

Figure 5 a,b: Mean and Second-Order moment of vertical displacement $U_5(t)$; Continuous line proposed method, Dots Monte-Carlo Simulation

Two different kind of time-varying amplitude has been reported: *i)* Harmonically time-varying excitation with different frequencies $f(t)=\sin(\Omega\, t)$; *ii)* An impulsive load $\delta(t)$

In fig.(4a, b) the mean $E\left[U_1(t)\right]$ and the second-order moment $E\left[U_1(t)^2\right]$ of the displacement function $u_1(t)$ has been. The statistics obtained with the proposed method have been contrasted with the corresponding estimates via Monte-Carlo simulation. The observation of fig.(4 a,b) shows that such a dynamical system is very sensitive to the presence of initial condition since the maxima of the response statistics have been obtained in the range $0-5.0$ sec and they are larger than stationary values.
Similar conclusions may be withdrawn from the observation of figs.(5 a,b) reporting, respectively, the mean $E\left[U_5(t)\right]$ and the second-order moment $E\left[U_5(t)^2\right]$ of the vertical displacement of node P_2 .

The statistics of the response due to the impulsive load have been reported in fig.(6 a,b) in which first-order statistics $E\left[U_1(t)\right], E\left[U_4(t)\right]$ have been contrasted with Monte-Carlo estimates (fig. 6 a, b). Second-order expectations $E\left[U_1(t)^2\right], E\left[U_4(t)^2\right]$ have been also reported in fig.(7 a, b) contrasting the Monte-Carlo estimates used as benchmark.

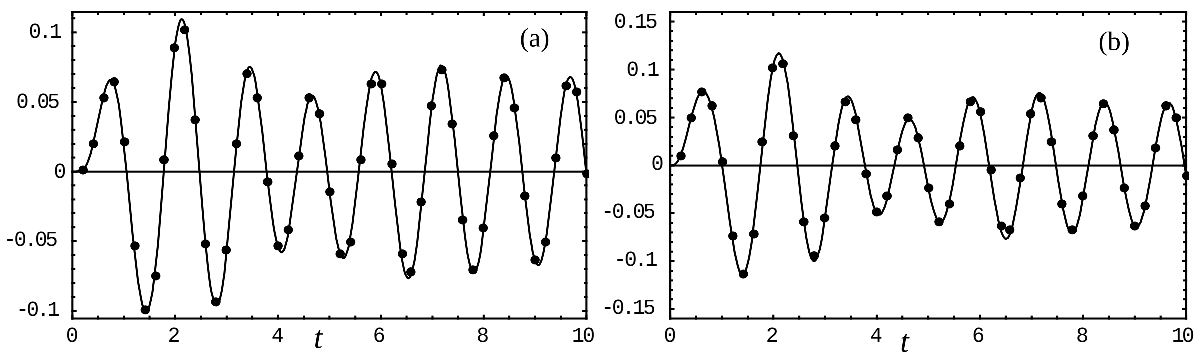

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Figure 6 a, b: Mean of horizontal displacement $U_1(t)$ and vertical displacements $U_4(t)$; Continuous line proposed method, Dots Monte-Carlo Simulation

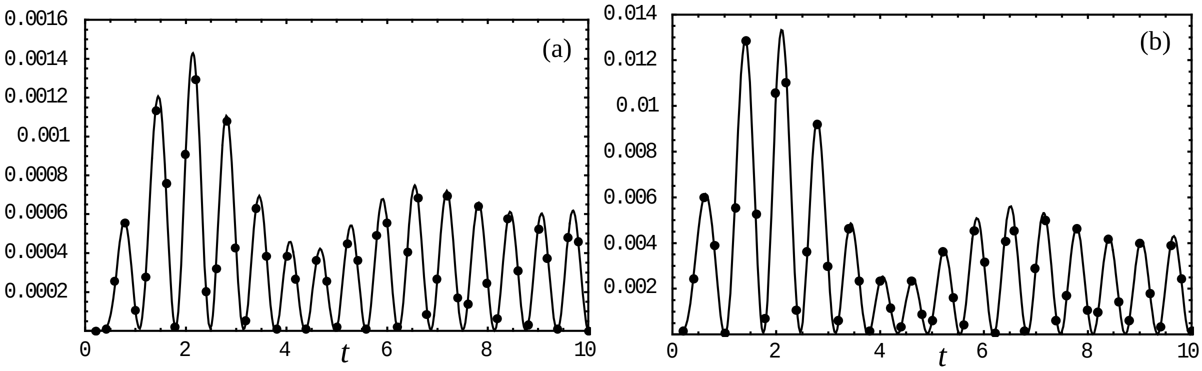

Figure 7 a, b: Second-Order moment of horizontal displacement $U_1(t)$ and vertical displacements $U_4(t)$; Continuous line proposed method, Dots Monte-Carlo Simulation

5 CONCLUSIONS

In this paper the non-stationary analysis of random structures, with random fluctuations of mechanical and/or geometrical parameters with respect to the nominal values, has been framed in the context of the Virtual Distortion Method (VDM). Analysis has been focused on random trusses to illustrate the capability of the VDM. Such an approach yields the structural displacement vector considering a deterministic structures under the external, time-varying load with a convenient set of superimposed virtual distortions depending on the random fluctuations as well as on the time varying axial stress. In the context of linear elasticity the structural response has been obtained resorting to Laplace transform yielding the displacement vector in the form of an asymptotic expansion that involves the random structural parameters. A convenient manipulation of the expansion by means of Kronecker algebra leads to an explicit expression of the structural response in terms of random parameter vector so that the statistics of the response may be evaluated in closed-form. It has been proved that every-order statistics of the random parameters are involved in the analysis of random structures and the provided formulation allows to take into account any order moment of random parameters that is important for the statistics of the structural response. A numerical study of an engineering-type truss has been provided to challenge the proposed formulation with the Monte-Carlo estimates of the response used as benchmark solutions for different kind of external loads. It has been observed that retaining two terms of the proposed expansion yields sufficiently accurate results with respect to Monte-Carlo estimates.

The proposed approach to dynamic analysis of random trusses may be also applied for random vibrations in presence of non-stationary random load and such an extension of the proposed approach has been reported in appendix.

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APPENDIX A: FUNDAMENTALS OF KRONECKER ALGEBRA

The Kronecker product between two matrices $\mathbf{A}$ and $\mathbf{B}$, respectively, of order $(m\times n)$ and $(p\times q)$ is a block matrix $\mathbf{C}$ of order $(m\cdot p\times n\cdot q)$ where each block is obtained multiplying each element a_{ij} of matrix $\mathbf{A}$ by the entire matrix $\mathbf{B}$, which reads:

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$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ \dots & \dots & \dots \\ a_{m1}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix} \tag{B1}$$

The Kronecker power $\mathbf{A}^{[k]}$ is defined as:

$$\mathbf{A}^{[k]} = \underbrace{\mathbf{A} \otimes \mathbf{A} \otimes \cdots \otimes \mathbf{A}}_{k-fold} \tag{B2}$$

Kronecker product has the following properties:

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}) \tag{B3}$$

$$(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T \tag{B4}$$

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1} \tag{B5}$$

provided that previous products exist.

The Kronecker sum of two matrices $\mathbf{A}$ and $\mathbf{B}$, respectively of order $(m \times m)$ and $(p \times p)$ is defined as:

$$\mathbf{C} = \mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{I}_m + \mathbf{I}_p \otimes \mathbf{B}$$

(B6) where $\mathbf{I}_m$ and $\mathbf{I}_p$ are, respectively, identity matrix of order $(m \times m)$ and $(p \times p)$. More details on Kronecker algebra may be found in [22].

APPENDIX B: RANDOM VIBRATIONS UNDER NON-STATIONARY GAUSSIAN EXCITATION

The proposed method to deal with random structures may be extented to the case of Gaussian random excitation. To this aim let us assume that external loads applied to the structure are represented by a zero-mean, Gaussian univariate random vector, indicated in the following by capital letter $\left(\mathbf{f}(t)\to\mathbf{F}(t)\right)$. Under this assumption vector of random processes $\mathbf{F}(t)$ can be represented as $\mathbf{F}(t)=\overline{\mathbf{F}}F(t)$ where $\overline{\mathbf{F}}$ is a $(N\times 1)$ deterministic vector and $F(t)$ is a zero-mean, non-stationary Gaussian process with prescribed statistics, namely the mean and second-order correlation, respectively:

$$\begin{aligned}&\mu_F(t)=E\left[F(t)\right]\\&R_F(t_1,t_2)=E\left[F(t_1)F(t_2)\right]-\mu_F(t_1)\mu_F(t_2)\end{aligned} \qquad \text{(B1 a, b)}$$

The analysis of the structural response in Laplace domain involves the Laplace integral transform of the random process $\hat{F}(s)=\wp\left[F(t)\right]$ which may be defined if the following condition holds:

$$\int_0^\infty\int_0^\infty R_F(t_1,t_2)e^{-s_1t_1}e^{-s_2t_2}dt_1dt_2<\infty \qquad \text{(B2)}$$

This latter requirement is fulfilled for the forcing random process $F(t)$ assuming that the autocorrelation function $R_F(t_1,t_2)$ vanishes asymptotically for $t_1,t_2\to\pm\infty$.
In the following we will assume that the uncertain structural parameters B_j $(j=1,2,...,N)$ belong to a symmetric and closed interval as in sec.3.

Similar arguments leading to the displacement vector in Laplace domain in sec.3 yields to express the displacement vector in Laplace space as:

$$\hat{\mathbf{u}}(s,B)=\hat{\mathbf{\Phi}}(s)\left(\sum_{j=\mathbf{0}}^{\infty}(-1)^j\,\mathbf{A}_e^{(j)}(s)\mathbf{b}^{[j]}\otimes\mathbf{I}\right)\hat{\mathbf{\Phi}}(s)^T\,\overline{\mathbf{F}}\hat{F}(s) \qquad \text{(B3)}$$

yielding, after mathematical expectations, the first and second-order vector moments of the displacement function in the form:

$$\hat{\boldsymbol{\mu}}_{\mathbf{u}}(s)=E\left[\hat{\mathbf{u}}(s,B)\right]=\hat{\mathbf{\Phi}}(s)\left(\sum_{j=\mathbf{0}}^{\infty}(-1)^j\,\mathbf{A}_e^{(j)}(s)\mathbf{m}_j(\mathbf{b})\otimes\mathbf{I}\right)\hat{\mathbf{\Phi}}(s)^T\,\overline{\mathbf{F}}\hat{\mu}_F(s) \qquad \text{(B4 a)}$$

$$\begin{aligned}&E\left[\hat{\mathbf{u}}(s_1,B)\otimes\hat{\mathbf{u}}(s_2,B)\right]\\&=\hat{\mathbf{\Phi}}^{(2)}(s_1,s_2)\left(\sum_{j=\mathbf{0}}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\,\mathbf{A}_e^{(j,k)}(s_1,s_2)\mathbf{m}_{j+k}(\mathbf{b})\otimes\mathbf{I}^{[2]}\right)\hat{\mathbf{\Phi}}^{(2)}(s_1,s_2)^T\,\overline{\mathbf{F}}^{[2]}\hat{R}_F(s_1,s_2)\end{aligned} \qquad \text{(B4 b)}$$

where we used the equivalence $E\left[\hat{F}(s)\right]=\hat{\mu}_F(s)$ and we introduced the expanded vectors:

$$\begin{aligned}&\mathbf{A}_e^{(j,k)}(s_1,s_2)=\mathbf{A}_e^{(j)}(s_1)\otimes\mathbf{A}_e^{(k)}(s_2)\\&\hat{\mathbf{\Phi}}^{(2)}(s_1,s_2)=\hat{\mathbf{\Phi}}(s_1)\otimes\hat{\mathbf{\Phi}}(s_2)\end{aligned} \tag{B5}$$

Time-domain evolution of the statistics of the nodal displacements may be achieved performing inverse Laplace transform defined in eqs.(B4 a, b) as:

$$\boldsymbol{\mu}_{\mathbf{u}}(t)=\wp^{-1}\left[\hat{\mathbf{\Phi}}(s)\left(\sum_{j=0}^{\infty}(-1)^j\,\mathbf{A}_e^{(j)}(s)\mathbf{m}_j(\mathbf{b})\otimes\mathbf{I}\right)\hat{\mathbf{\Phi}}(s)^T\,\overline{\mathbf{F}}\hat{F}(s)\right] \tag{B6 a}$$

$$\begin{aligned}&E\left[\mathbf{u}(t_1,B)\otimes\mathbf{u}(t_2,B)\right]\\&=\wp_2^{-1}\left[\hat{\mathbf{\Phi}}^{(2)}(s_1,s_2)\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\,\mathbf{A}_e^{(j,k)}(s_1,s_2)\mathbf{m}_{j+k}(\mathbf{b})\otimes\mathbf{I}^{[2]}\right)\hat{\mathbf{\Phi}}^{(2)}(s_1,s_2)^T\,\mathbf{F}_{\mathbf{0}}^{[2]}R_F(s_1,s_2)\right]\end{aligned} \tag{B6 b}$$

holding in presence of zero-mean random process. Observation of eq.(B6 b) shows that second-order moments in time domain in presence of random loads is achieved by two-dimensional inverse Laplace transform in eq.(B4 b) since correlation function in the latter factor couples Laplace parameters s_1 and s_2.