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EXACT STATIC DEFLECTION OF NON-UNIFORM EULER-BERNOULLI BEAMS WITH FLEXURAL STIFFNESS SINGULARITIES

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Abstract. *In this study the theory of distributions is adopted to find closed form solutions of the static governing equations of non-uniform Euler Bernoulli beams. In particular, the solutions are obtained as limit cases of multi-step uniform beams. According to the proposed approach, non-uniform beams in presence of an arbitrary number of singularities modelled as unit step functions or Dirac's deltas can also be treated. The latter cases provide closed form solutions of non-uniform beams, whose flexural stiffness is not differentiable, showing discontinuities in the curvature and the slope functions, respectively.*

Sommario. *Questo lavoro considera la teoria delle distribuzioni allo scopo di determinare la soluzione esatta in forma chiusa delle equazioni della trave di Eulero-Bernoulli in regime statico. In particolare, le soluzioni sono ottenute come caso limite di una trave soggetta a discontinuità multiple. Mediante l'approccio proposto sono anche trattate travi non-uniformi in presenza di un numero arbitrario di singolarità modellate con distribuzioni gradino unitario e delta di Dirac. Il caso di singolarità modellate con delta di Dirac fornisce soluzioni in forma chiusa di travi non-uniformi la cui rigidezza flessionale non è una funzione differenziabile e conduce a discontinuità sia nelle curvature che nelle rotazioni.*

1 INTRODUCTION

Engineering applications requiring the adoption of non-uniform beams aim at different aspects such as optimization of weight distribution, architectural and functional tasks, improvement of internal stress distribution, etc. Therefore, the study of procedures for the analysis of beams with variable flexural stiffness can be of great interest in mechanical, aeronautical and structural engineering fields.

The governing equations of non-uniform beams subjected to static loads were originally treated by means of iterative procedures and are reported in standard text books[1,2]. However, analytical solutions have been proposed in the literature by several authors under the hypothesis that the variable flexural stiffness is second-order differentiable[3-7]. Some of the latter procedures regard particular boundary conditions, others are more general and require the knowledge of fundamental solutions. In any case, generalization to non differentiable variable flexural stiffness is not allowed even in presence strong singularities.

The presence of discontinuities superimposed onto a non-uniform flexural stiffness has been treated in the past in the space of the generalized functions (distributions)[8]. In particular, the bending of non-uniform beams with jump discontinuities has been formulated without the need of partitioning the beam into continuous segments; however, besides the boundary conditions, enforcement of continuity at each singularity is yet required.

Aim of this work is providing the exact deflection function for non-uniform beams with flexural stiffness models which are second-order differentiable and subsequently generalizing the solution in presence of flexural stiffness singularities without enforcement of any continuity condition.

Recently the authors treated the case of uniform flexural stiffness in presence of single and multiple singularities by making use of the distribution theory, and provided closed form solutions for different types of singularities[9,10]. In this paper the latter closed form solutions are considered and generalized for the case of non-uniform beams with non-differentiable flexural stiffness. In particular, it is shown, first, that the model showing abrupt changes in the flexural stiffness can be conveniently adopted to treat multi-step beams. Hence, the solution of non-uniform beams is obtained as the limit case where the number of flexural stiffness discontinuities tends to infinity. The explicit solution for non-uniform beams is proposed in integral form and is shown to hold in presence of flexural stiffness discontinuities.

Furthermore, the model of uniform beams in presence of flexural stiffness discontinuities and slope discontinuities is considered to obtain closed form solutions of non-uniform beams with internal hinges.

The presented closed form solutions are adopted to provide explicit expressions for different non-uniform beams with different external load functions. A numerical application to a non-uniform beam with a non-differentiable flexural stiffness due to the presence of different singularities is presented. Finally, the case of a beam showing non-uniform flexural stiffness distributions due to the presence of concentrated cracks is also analysed.

2 GOVERNING EQUATIONS OF THE NON-UNIFORM EULER-BERNOULLI BEAM

The governing equations of the Euler-Bernoulli beam are written as follows:

$$V^{I}(x) = -q(x) \quad , \quad M^{I}(x) = V(x) \quad ,$$
$$\chi(x) = \frac{M(x)}{E(x)I(x)} \quad , \tag{1}$$
$$\chi(x) = \varphi^{I}(x) \quad , \quad \varphi(x) = -u^{I}(x) \quad ,$$

where $q(x)$ is the external transversal load, $V(x)$ and $M(x)$ are the shear force and the bending moment, respectively, $u(x)$, $\varphi(x)$ and $\chi(x)$ are the deflection, slope and curvature functions, respectively, $E(x)$ and $I(x)$ are the Young's modulus and the inertia moment, respectively, and the prime denotes differentiation with respect to the spatial coordinate x spanning from 0 to the length L of the beam.

Combining the equilibrium, constitutive and compatibility equations given by Eqs. (1) yields to the following fourth order differential equation:

$$\left[E(x)I(x)u''(x)\right]'' = q(x) \quad . \tag{2}$$

Eq. (2) is the governing differential equation of the Euler-Bernoulli beam with variable flexural stiffness $E(x)I(x)$.

In this study, in order to propose an integration procedure of Eq.(2) for any flexural stiffness function, the following piecewise constant flexural stiffness model is adopted:

$$\left[E(x)I(x)\right]_{pw} = E(0)I(0)\left[1-\sum_{i=1}^{n}\gamma_i U\left(x-x_{\gamma,i}\right)\right] \tag{3}$$

where $U\left(x-x_{\gamma,i}\right)$ is the unit step distribution, also known in the literature as Heaviside's function, showing a discontinuity at the abscissa $x_{\gamma,i}$ and defined as: $U\left(x-x_{\gamma,i}\right)=0$ for $x<x_{\gamma,i}$, and $U\left(x-x_{\gamma,i}\right)=1$ for $x>x_{\gamma,i}$. Furthermore, the scalar parameters γ_i, $i=1,\ldots,n$, appearing in Eq. (3) provide the intensities of the flexural stiffness jumps at abscissae $x_{\gamma,i}$.

According to Eq. (3), by chosing abscissae $x_{\gamma,i}$ such that $x_{\gamma,i}-x_{\gamma,i-1}=\Delta x$ for $i=1,\ldots,n$, the beam results subdivided into $n+1$ uniform stubs of equal length: $\Delta x = L/(n+1)$ with constant flexural stiffness given by $E_iI_i = E(0)I(0)\left[1-\sum_{k=1}^{i-1}\gamma_k U\left(x-x_{\gamma,k}\right)\right]$, for $i=1,\ldots,n$, as depicted in Fig.1.

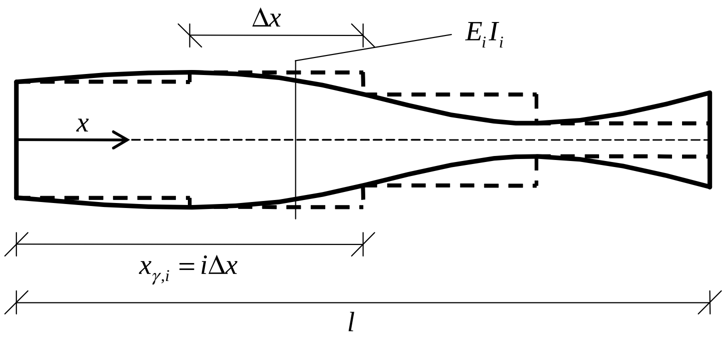

Figure 1: An approximate piecewise model of a non-uniform beam

It has to be noted that, for the flexural stiffnesses E_iI_i to be non negative, the only constraints to be imposed on jump intensities γ_i are: $\sum_{j=1}^{i}\gamma_j \le 1$, $i=1,\ldots,n$.

3 INTEGRATION PROCEDURE FOR NON-UNIFORM EULER-BERNOULLI BEAMS

In this section the flexural stiffness model adopted in Eq. (3) is considered for integration of the governing differential equation (2) and it will be shown how the explicit solution concerning beams with continuously varying flexural stiffness can be inferred.

Let us consider a non uniform Euler-Bernoulli beam, with variable flexural stiffness $E(x)I(x)$, governed by Eq. (2). An approximate expression of the flexural stiffness $E(x)I(x)$ can be given by the piecewise model in Eq. (3) such that at the abscissas $x = x_{\gamma,i}$ the exact and approximate flexural stiffness assume the same value $E(x_{\gamma,i})I(x_{\gamma,i})$. In view of the adopted flexural stiffness model, the following positions can be accounted for:

$$\begin{gathered} x_{\gamma,i} = i\,\Delta x \quad , \\ E(0)I(0) = E_0 I_0 \quad , \quad E(x_{\gamma,i})I(x_{\gamma,i}) = E_i I_i \quad , \\ \gamma_i = \frac{E_{i-1}I_{i-1} - E_i I_i}{E_0 I_0} \quad . \end{gathered} \tag{4}$$

The limit of Eq. (3) for $n \to \infty$ can be written as follows:

$$\begin{aligned} \lim_{n\to\infty}\left[E(x)I(x)\right]_{pw} &= E_0 I_0 - \lim_{n\to\infty}\sum_{i=1}^{n}\frac{E_{i-1}I_{i-1} - E_i I_i}{\Delta x} U\left(x - x_{\gamma,i}\right)\Delta x = \\ &= E(0)I(0) + \int_0^l \left[E(\xi)I(\xi)\right]^{\prime} U\left(x-\xi\right) d\xi = E(x)I(x) \end{aligned} \tag{5}$$

where Eqs. (4a-c) have been accounted for. According to Eq. (5), the flexural stiffness $E(x)I(x)$ is recovered as the limit of the piecewise model for $n \to \infty$ hence for $\Delta x \to 0$.

In view of the property obtained in Eq. (5) the explicit solution of the Euler-Bernoulli beam with the approximate model given by Eq. (3) can lead to the solution of the non uniform beam.

For the adopted approximate expression of the flexural stiffness given by Eq. (3), the governing equation (2) assumes the following form:

$$\left[E_0 I_0 \left(1 - \sum_{i=1}^{n}\gamma_i U\left(x - x_{\gamma,i}\right)\right) u_{pw}^{II}(x)\right]^{II} = q(x) \tag{6}$$

where $u_{pw}(x)$ is the approximate deflection function of the beam.

As recently shown in the literature[10],the closed form solution of Euler-Bernoulli beams subjected to abrupt changes of the flexural stiffness are governed by Eq. (6) and the following closed form expression of the approximate deflection function can be obtained:

$$u_{pw}(x) = c_1 + c_2 x + c_3 \left[x^2 + \sum_{i=1}^{n} \gamma_i \mu_i \mu_{i+1} \left(x - x_{\gamma,i} \right)^2 U\left(x - x_{\gamma,i} \right) \right] + \\ + c_4 \left[x^3 + \sum_{i=1}^{n} \gamma_i \mu_i \mu_{i+1} \left(x^3 - 3x_{\gamma,i}^2 x + 2x_{\gamma,i}^3 \right) U\left(x - x_{\gamma,i} \right) \right] + \\ + \frac{q^{[4]}(x)}{E_0 I_0} + \sum_{i=1}^{n} \gamma_i \mu_i \mu_{i+1} \frac{q^{[4]}(x) - q^{[4]}(x_{\gamma,i}) - q^{[3]}(x_{\gamma,i})\left(x - x_{\gamma,i} \right)}{E_0 I_0} U\left(x - x_{\gamma,i} \right) \quad . \tag{7}$$

In Eq. (7) c_1, c_2, c_3, c_4 are integration constants, $q^{[k]}(x)$ denotes a function evaluated as a primitive of order *k* of the external transversal load function $q(x)$ and the following position have been accounted for:

$$\mu_i = \frac{1}{1 - \sum_{j=1}^{i-1} \gamma_j} = \frac{E_0 I_0}{E_{i-1} I_{i-1}} \quad . \tag{8}$$

According to Eq. (5) the non uniform flexural stiffness has been obtained as the limit of a discontinuous model, hence, in view of the linearity of the problem, the deflection function $u(x)$ of the non-uniform beam with flexural stiffness $E(x)I(x)$ can be obtained from the approximate deflection function $u_{pw}(x)$ by taking the limit for $n \to \infty$ of Eq. (7) as follows:

$$u(x) = \lim_{n\to\infty} u_{pw}(x) = c_1 + c_2 x + c_3 \left[x^2 + E_0 I_0 \lim_{n\to\infty} \sum_{i=1}^{n} \frac{E_{i-1} I_{i-1} - E_i I_i}{\left(E_{i-1} I_{i-1} \right)\left(E_i I_i \right)} \left(x - x_{\gamma,i} \right)^2 U\left(x - x_{\gamma,i} \right) \right] + \\ + c_4 \left[x^3 + E_0 I_0 \lim_{n\to\infty} \sum_{i=1}^{n} \frac{E_{i-1} I_{i-1} - E_i I_i}{\left(E_{i-1} I_{i-1} \right)\left(E_i I_i \right)} \left(x^3 - 3x_{\gamma,i}^2 x + 2x_{\gamma,i}^3 \right) U\left(x - x_{\gamma,i} \right) \right] + \\ + \frac{q^{[4]}(x)}{E_0 I_0} + \lim_{n\to\infty} \sum_{i=1}^{n} \frac{E_{i-1} I_{i-1} - E_i I_i}{\left(E_{i-1} I_{i-1} \right)\left(E_i I_i \right)} \left[q^{[4]}(x) - q^{[4]}(x_{\gamma,i}) - q^{[3]}(x_{\gamma,i})\left(x - x_{\gamma,i} \right) \right] U\left(x - x_{\gamma,i} \right) \tag{9}$$

where the positions given by Eqs. (4) and (8) have been accounted for.

By solving the limits appearing in Eq. (9) the following closed form of the deflection function of the non-uniform Euler-Bernoulli beam is obtained:

$$u(x) = c_1 + c_2 x + c_3 \left(x^2 - E(0)I(0) \int_0^l \frac{\left[E(\xi)I(\xi) \right]'}{\left(E(\xi)I(\xi) \right)^2} \left(x - \xi \right)^2 U\left(x - \xi \right) d\xi \right) + \\ + c_4 \left(x^3 - E(0)I(0) \int_0^l \frac{\left[E(\xi)I(\xi) \right]'}{\left(E(\xi)I(\xi) \right)^2} \left(x^3 - 3\xi^2 x + 2\xi^3 \right) U\left(x - \xi \right) d\xi \right) + \\ + \frac{q^{[4]}(x)}{E(0)I(0)} - \int_0^l \frac{\left[E(\xi)I(\xi) \right]'}{\left(E(\xi)I(\xi) \right)^2} \left(q^{[4]}(x) - q^{[4]}(\xi) - q^{[3]}(\xi)\left(x - \xi \right) \right) U\left(x - \xi \right) d\xi \tag{10}$$

Eq. (10) can be further simplified, by means of integration by parts, as follows:

$$u(x)=c_1+c_2x+2c_3E(0)I(0)\int_0^x \frac{x-\xi}{E(\xi)I(\xi)}d\xi+6c_4E(0)I(0)\int_0^x \frac{(x-\xi)\xi}{E(\xi)I(\xi)}d\xi+ \\ +\frac{q^{[4]}(0)+q^{[3]}(0)x}{E(0)I(0)}+\int_0^x \frac{x-\xi}{E(\xi)I(\xi)}q^{[2]}(\xi)d\xi \quad . \tag{11}$$

Subsequent differentiations of the deflection function given by Eq. (11) provide the following closed form expression for the slope and curvature functions of the non-uniform beam, respectively:

$$\varphi(x)=-u'(x)=-c_2-2c_3E(0)I(0)\int_0^x \frac{1}{E(\xi)I(\xi)}d\xi-6c_4E(0)I(0)\int_0^x \frac{\xi}{E(\xi)I(\xi)}d\xi- \\ -\frac{q^{[3]}(0)}{E(0)I(0)}-\int_0^x \frac{q^{[2]}(\xi)}{E(\xi)I(\xi)}d\xi \quad , \\ \chi(x)=-u''(x)=-\left(2c_3+6c_4x+\frac{q^{[2]}(x)}{E(0)I(0)}\right)\frac{E(0)I(0)}{E(x)I(x)} \quad . \tag{12}$$

It has to be noted that Eqs. (12), representing the slope and curvature of the non-uniform beam can also be recovered by taking the limit for $n \to \infty$ of the corresponding approximate functions obtained by differentiating the approximate deflection function given by Eq. (7).
The bending moment function is obtained by multiplying the curvature function given by Eq. (12 b) by the continuously variable flexural stiffness $E(x)I(x)$ as follows:

$$M(x)=E(x)I(x)\chi(x)=-E(0)I(0)\left(2c_3+6c_4x+\frac{q^{[2]}(x)}{E(0)I(0)}\right) \quad . \tag{13}$$

Finally, the shear force function is obtained by means of differentiation of Eq. (13) as follows:

$$V(x)=M'(x)=-E(0)I(0)\left(6c_4+\frac{q^{[1]}(x)}{E(0)I(0)}\right) \quad . \tag{14}$$

It has to be remarked that, since the solution of non-uniform beams presented in this section has been obtained as the limit of a beam with discontinuous flexural stiffness, Eq. (11) holds also for non-uniform beams where the flexural stiffness is non-differentiable due to the presence of flexural stiffness discontinuities, such as abrupt changes of the material or the cross-section.

4 THE NON-UNIFORM EULER-BERNOULLI BEAM IN PRESENCE OF SLOPE DISCONTINUITIES

The piecewise flexural stiffness model adopted in Eq. (3) has been shown in the previous section to be useful in order to obtain the explicit solution of non-uniform beams. In this section the piecewise model introduced in Eq.(3) in enriched by means of additional distributions such as Dirac's deltas able to represent discontinuities in the slope function and it will be shown how the explicit expressions of the response functions of non-uniform beams in presence of slope discontinuities are obtained.

The piecewise flexural stiffness model adopted in Eq. (3) to describe abrupt changes of the flexural stiffness can be enriched by means of the superposition of Dirac's delta distributions centred at abscissae $x_{\beta,j}$, $j=1,\ldots,m$, such that $x_{\beta,j}\neq x_{\gamma,i}$, $\forall\, i,j$, as follows:

$$\left[E(x)I(x)\right]_{pw}=E_0I_0\left[1-\sum_{i=1}^{n}\gamma_i U\left(x-x_{\gamma,i}\right)-\sum_{j=1}^{m}\beta_j\delta\left(x-x_{\beta,j}\right)\right] \quad . \tag{15}$$

The introduction of Dirac's deltas into a uniform flexural stiffness model has been recently shown to be equivalent to the presence of internal hinges endowed with rotational springs hence leading to discontinuities in the slope function[9].

In this case, the governing equation of a non uniform Euler-Bernoulli beam, with variable flexural stiffness $E(x)I(x)$ in presence of concentrated slope discontinuities can be approximated as follows:

$$\left[E_0I_0\left(1-\sum_{i=1}^{n}\gamma_i U\left(x-x_{\gamma,i}\right)-\sum_{j=1}^{m}\beta_j\delta\left(x-x_{\beta,j}\right)\right)u_{pw}^{II}(x)\right]^{II}=q(x) \quad . \tag{16}$$

The explicit solution of the governing equation (16) has been obtained in a recent work by employing distribution integration rules involving the product of Diracs' deltas[10] and takes the following form:

$$\begin{aligned}u_{pw}(x)=&\,c_1+c_2x+c_3\left[x^2+\sum_{i=1}^{n}\overline{\gamma}_i\left(x-x_{\gamma,i}\right)^2U\left(x-x_{\gamma,i}\right)+2l\sum_{j=1}^{m}\overline{\beta}_j\left(x-x_{\beta,j}\right)U\left(x-x_{\beta,j}\right)\right]+\\&+c_4\left[x^3+\sum_{i=1}^{n}\overline{\gamma}_i\left(x^3-3x_{\gamma,i}^2x+2x_{\gamma,i}^3\right)U\left(x-x_{\gamma,i}\right)+6l\sum_{j=1}^{m}\overline{\beta}_j x_{\beta,j}\left(x-x_{\beta,j}\right)U\left(x-x_{\beta,j}\right)\right]+\\&+\frac{q^{[4]}(x)}{E_0I_0}+\sum_{i=1}^{n}\overline{\gamma}_i\frac{q^{[4]}(x)-q^{[4]}(x_{\gamma,i})-q^{[3]}(x_{\gamma,i})\left(x-x_{\gamma,i}\right)}{E_0I_0}U\left(x-x_{\gamma,i}\right)+\\&+l\sum_{j=1}^{m}\overline{\beta}_j\frac{q^{[2]}(x_{\beta,j})\left(x-x_{\beta,j}\right)}{E_0I_0}U\left(x-x_{\beta,j}\right)\end{aligned} \tag{17}$$

where:

$$\overline{\gamma}_i=\gamma_i\mu_i\mu_{i+1} \quad , \tag{18}$$

$$\overline{\beta}_j=\frac{1}{l}\left(\frac{\beta_j}{1-\beta_jA}+\eta_j\right)\left(1+\sum_{i=1}^{n}\overline{\gamma}_iU\left(x_{\beta,j}-x_{\gamma,i}\right)\right) \quad , \tag{19}$$

$$\eta_j=\beta_j\sum_{i=1}^{n}\frac{\overline{\gamma}_i}{\left(1-\mu_i\beta_jA\right)\left(1-\mu_{i+1}\beta_jA\right)}U\left(x_{\beta,j}-x_{\gamma,i}\right) \quad , \tag{20}$$

and μ_i is given by Eq. (8).

The constant A appearing in Eqs.(19),(20) arises from the definition of the product of two Dirac's delta distributions proposed by Bagarello[11,12]. Bagarello indicates that the product of two Dirac's deltas both centred at x_0 can be reduced to a single Dirac's delta multiplied by a constant A. A set of values that can be adopted for the quantity A, is reported in the Appendix of the paper[10].

For the sake of completeness, the stiffnesses $k_{\varphi,j}$ of the rotational springs equivalent to the Dirac's deltas are here reported as follows: $k_{\varphi,j} = E_0 I_0 /(l\overline{\beta}_j)$.

Furthemore, besides the relationships introduced in Eqs. (4) still holding, the following relationships between $\overline{\gamma}_i$, $\overline{\beta}_j$, η_j and the values of the continuous flexural stiffness function $E(x)I(x)$ hold:

$$\overline{\gamma}_i = E_0 I_0 \frac{E_{i-1}I_{i-1} - E_i I_i}{\left(E_{i-1}I_{i-1}\right)\left(E_i I_i\right)} \quad , \tag{21}$$

$$\eta_j = E_0 I_0 \beta_j \sum_{i=1}^{n} \frac{E_{i-1}I_{i-1} - E_i I_i}{\left(E_{i-1}I_{i-1} - E_0 I_0 \beta_j A\right)\left(E_i I_i - E_0 I_0 \beta_j A\right)} U\left(x_{\beta,j} - i\Delta x\right) \quad , \tag{22}$$

$$\overline{\beta}_j = \frac{1}{l}\left(\frac{\beta_j}{1-\beta_j A} + \eta_j\right)\left(1 + E_0 I_0 \sum_{i=1}^{n} \frac{E_{i-1}I_{i-1} - E_i I_i}{\left(E_{i-1}I_{i-1}\right)\left(E_i I_i\right)} U\left(x_{\beta,j} - i\Delta x\right)\right) \quad . \tag{23}$$

The deflection function $u(x)$ of the non-uniform beam with non-uniform flexural stiffness $E(x)I(x)$ in presence of internal hinges can be obtained from the approximate deflection function $u_{pw}(x)$, given by Eq. (17), by taking the limit for $n \to \infty$, as follows:

$$\begin{aligned}
u(x) &= c_1 + c_2 x + \\
&+ c_3\left[x^2 + E_0 I_0 \lim_{n\to\infty}\sum_{i=1}^{n} \frac{E_{i-1}I_{i-1} - E_i I_i}{\left(E_{i-1}I_{i-1}\right)\left(E_i I_i\right)\Delta x}\left(x - x_{\gamma,i}\right)^2 U\left(x - x_{\gamma,i}\right)\Delta x + 2l\sum_{j=1}^{m}\overline{\beta}_j^{(\mathrm{lim})}\left(x - x_{\beta,j}\right)U\left(x - x_{\beta,j}\right)\right] \\
&+ c_4\left[x^3 + E_0 I_0 \lim_{n\to\infty}\sum_{i=1}^{n} \frac{E_{i-1}I_{i-1} - E_i I_i}{\left(E_{i-1}I_{i-1}\right)\left(E_i I_i\right)\Delta x}\left(x^3 - 3x_{\gamma,i}^2 x + 2x_{\gamma,i}^3\right)U\left(x - x_{\gamma,i}\right)\Delta x + \right. \\
&\left. + 6l\sum_{j=1}^{m}\overline{\beta}_j^{(\mathrm{lim})} x_{\beta,j}\left(x - x_{\beta,j}\right)U\left(x - x_{\beta,j}\right)\right] + \\
&+ \frac{q^{[4]}(x)}{E_0 I_0} + E_0 I_0 \lim_{n\to\infty}\sum_{i=1}^{n} \frac{E_{i-1}I_{i-1} - E_i I_i}{\left(E_{i-1}I_{i-1}\right)\left(E_i I_i\right)\Delta x}\frac{q^{[4]}(x) - q^{[4]}(x_{\gamma,i}) - q^{[3]}(x_{\gamma,i})\left(x - x_{\gamma,i}\right)}{E_0 I_0}U\left(x - x_{\gamma,i}\right)\Delta x + \\
&+ l\sum_{j=1}^{m}\overline{\beta}_j^{(\mathrm{lim})}\frac{q^{[2]}(x_{\beta,j})\left(x - x_{\beta,j}\right)}{E_0 I_0}U\left(x - x_{\beta,j}\right)
\end{aligned} \tag{24}$$

where:

$$\begin{aligned}
\overline{\beta}_j^{(\mathrm{lim})} &= \frac{1}{l}\left(\frac{\beta_j}{1-\beta_j A} + E_0 I_0 \beta_j \lim_{n\to\infty}\sum_{i=1}^{n} \frac{E_{i-1}I_{i-1} - E_i I_i}{\left(E_{i-1}I_{i-1} - E_0 I_0 \beta_j A\right)\left(E_i I_i - E_0 I_0 \beta_j A\right)\Delta x}U\left(x_{\beta,j} - i\Delta x\right)\Delta x\right)\cdot \\
&\cdot\left(1 + E_0 I_0 \lim_{n\to\infty}\sum_{i=1}^{n} \frac{E_{i-1}I_{i-1} - E_i I_i}{\left(E_{i-1}I_{i-1}\right)\left(E_i I_i\right)\Delta x}U\left(x_{\beta,j} - i\Delta x\right)\Delta x\right) \quad .
\end{aligned} \tag{25}$$

where the positions given by Eqs. (4), (8), (21)-(23) have been accounted for.

By solving the limits appearing in Eqs. (24) and (25) the following closed form of the deflection function of the non-uniform Euler-Bernoulli beam is obtained:

$$
\begin{aligned}
u(x) &= c_1 + c_2 x + \\
&+ c_3\left[x^2 - E(0)I(0)\int_0^l \frac{\left[E(\xi)I(\xi)\right]'}{\left(E(\xi)I(\xi)\right)^2}\left(x-\xi\right)^2 U\left(x-\xi\right)d\xi + 2l\sum_{j=1}^{m}\overline{\beta}_j^{(\lim)}\left(x-x_{\beta,j}\right)U\left(x-x_{\beta,j}\right)\right] \\
&+ c_4\left[x^3 - E(0)I(0)\int_0^l \frac{\left[E(\xi)I(\xi)\right]'}{\left(E(\xi)I(\xi)\right)^2}\left(x^3-3\xi^2 x+2\xi^3\right)U\left(x-\xi\right)d\xi + \right. \\
&\left. + 6l\sum_{j=1}^{m}\overline{\beta}_j^{(\lim)}x_{\beta,j}\left(x-x_{\beta,j}\right)U\left(x-x_{\beta,j}\right)\right] + \\
&+ \frac{q^{[4]}(x)}{E(0)I(0)} - \int_0^l \frac{\left[E(\xi)I(\xi)\right]'}{\left(E(\xi)I(\xi)\right)^2}\left(q^{[4]}(x) - q^{[4]}(\xi) - q^{[3]}(\xi)\left(x-\xi\right)\right)U\left(x-\xi\right)d\xi + \\
&+ l\sum_{j=1}^{m}\overline{\beta}_j^{(\lim)}\frac{q^{[2]}(x_{\beta,j})\left(x-x_{\beta,j}\right)}{E(0)I(0)}U\left(x-x_{\beta,j}\right)
\end{aligned}
\tag{26}
$$

where:

$$
\begin{aligned}
\overline{\beta}_j^{(\lim)} &= \frac{1}{l}\left(\frac{\beta_j}{1-\beta_j A} - E(0)I(0)\beta_j\int_0^l \frac{\left[E(\xi)I(\xi)\right]'}{\left(E(\xi)I(\xi) - E(0)I(0)\beta_j A\right)^2}U\left(x_{\beta,j}-\xi\right)d\xi\right)\cdot \\
&\cdot\left(1 - E(0)I(0)\int_0^l \frac{\left[E(\xi)I(\xi)\right]'}{\left(E(\xi)I(\xi)\right)^2}U\left(x_{\beta,j}-\xi\right)d\xi\right) \quad .
\end{aligned}
\tag{27}
$$

Eq. (26) can be further simplified, by means of integration by parts, as follows:

$$
\begin{aligned}
u(x) &= c_1 + c_2 x + \\
&+ 2c_3\left[E(0)I(0)\int_0^x \frac{x-\xi}{E(\xi)I(\xi)}d\xi + l\sum_{j=1}^{m}\overline{\beta}_j^{(\lim)}\left(x-x_{\beta,j}\right)U\left(x-x_{\beta,j}\right)\right] + \\
&+ 6c_4\left[E(0)I(0)\int_0^x \frac{\left(x-\xi\right)\xi}{E(\xi)I(\xi)}d\xi + l\sum_{j=1}^{m}\overline{\beta}_j^{(\lim)}x_{\beta,j}\left(x-x_{\beta,j}\right)U\left(x-x_{\beta,j}\right)\right] + \\
&+ \frac{q^{[4]}(0) + q^{[3]}(0)x}{E(0)I(0)} + \int_0^x \frac{x-\xi}{E(\xi)I(\xi)}q^{[2]}(\xi)d\xi + l\sum_{j=1}^{m}\overline{\beta}_j^{(\lim)}\frac{q^{[2]}(x_{\beta,j})\left(x-x_{\beta,j}\right)}{E(0)I(0)}U\left(x-x_{\beta,j}\right)
\end{aligned}
\tag{28}
$$

where:

$$
\overline{\beta}_j^{(\lim)} = \frac{1}{l}\frac{E(0)I(0)}{E(x_{\beta,j})I(x_{\beta,j})}\frac{E(0)I(0)\beta_j}{E(x_{\beta,j})I(x_{\beta,j}) - E(0)I(0)\beta_j A} \tag{29}
$$

Subsequent differentiations of the deflection function given by Eq. (28) provide the following closed form expression for the slope and curvature functions of the non-uniform beam with slope discontinuities, respectively:

$$\varphi(x)=-u^{I}(x)=-c_2-2c_3\left[E(0)I(0)\int_0^x\frac{1}{E(\xi)I(\xi)}d\xi+l\sum_{j=1}^{m}\overline{\beta}_j^{(\lim)}U\left(x-x_{\beta,j}\right)\right]-$$
$$-6c_4\left[E(0)I(0)\int_0^x\frac{\xi}{E(\xi)I(\xi)}d\xi+l\sum_{j=1}^{m}\overline{\beta}_j^{(\lim)}x_{\beta,j}U\left(x-x_{\beta,j}\right)\right]- \tag{30a}$$
$$-\frac{q^{[3]}(0)}{E(0)I(0)}-\int_0^x\frac{q^{[2]}(\xi)}{E(\xi)I(\xi)}d\xi-l\sum_{j=1}^{m}\overline{\beta}_j^{(\lim)}\frac{q^{[2]}(x_{\beta,j})}{E(0)I(0)}U\left(x-x_{\beta,j}\right)\quad,$$

$$\chi(x)=-u^{II}(x)=-\left(2c_3+6c_4x+\frac{q^{[2]}(x)}{E(0)I(0)}\right)\left(\frac{E(0)I(0)}{E(x)I(x)}+l\sum_{j=1}^{m}\overline{\beta}_j^{(\lim)}\delta\left(x-x_{\beta,j}\right)\right)\quad. \tag{30b}$$

The bending moment function is obtained by multiplying the curvature function given by Eq. (30 b) by the continuously variable flexural stiffness $E(x)I(x)$ in presence of singularities, and the shear force function is obtained by means of differentiation of the bending moment function. For the bending moment and the shear force functions the same expressions provided by Eqs. (13) and (14) concerning the case of beam without singularities are obtained. In fact, for statically determinate beams the bending moment and the shear force should not depend on the adopted flexural stiffness. On the contrary, for statically determinate beams, the adopted flexural stiffness model with Dirac's delta singularities will affect the expressions of the constants c_3, c_4 appearing in Eqs. (13), (14).

5 PARTICULAR CLOSED FORM SOLUTIONS

In this section the closed form solutions presented in the previous sections for non-uniform beams in presence of flexural stiffness and slope discontinuities are particularized for different variation laws of the flexural stiffness in order to show how any case can be easily treated without difficulties concerning with the integration procedure.

For the case of parabolic variable distributed load the following primitive functions of the external load are considered:

$$q(x)=q_0\left(1+\alpha_1x+\alpha_2x^2\right)$$
$$q^{[1]}(x)=\frac{q_0}{6}\left(6x+3\alpha_1x^2+2\alpha_2x^3\right)\quad,\quad q^{[2]}(x)=\frac{q_0}{12}\left(6x^2+2\alpha_1x^3+\alpha_2x^4\right) \tag{31}$$
$$q^{[3]}(x)=\frac{q_0}{120}\left(20x^3+5\alpha_1x^4+2\alpha_2x^5\right)\quad,\quad q^{[4]}(x)=\frac{q_0}{360}\left(15x^4+3\alpha_1x^5+\alpha_2x^6\right)$$

where l denote the beam length. The load parameters q_0, α_1 and α_2 appearing in Eqs. (31) can also be chosen to obtain a uniformly distributed load ($q_0\neq0$, $\alpha_1=\alpha_2=0$) and a linearly distributed load ($q_0\neq0$, $\alpha_1\neq0$, $\alpha_2=0$).

For the case of n_p point loads P_k concentrated at abscissae x_k, $k=1,\ldots.n_p$, the following primitive functions of the external load are considered:

$$q(x)=\sum_{k=1}^{n_p} P_k\,\delta(x-x_k)$$
$$q^{[1]}(x)=\sum_{k=1}^{n_p} P_k\,U(x-x_k)\quad,\quad q^{[2]}(x)=\sum_{k=1}^{n_p} P_k\,(x-x_k)U(x-x_k)\quad, \tag{32}$$
$$q^{[3]}(x)=\sum_{k=1}^{n_p} P_k\,\frac{(x-x_k)^2}{2}U(x-x_k)\quad,\quad q^{[4]}(x)=\sum_{k=1}^{n_p} P_k\,\frac{(x-x_k)^3}{6}U(x-x_k)\quad.$$

For the case of n_M concentrated moments M_s at abscissae x_s, $s=1,\ldots .n_M$, the following primitive functions of the external load are considered:

$$q(x)=\sum_{s=1}^{n_M} M_s\,\delta^{[1]}(x-x_s)$$
$$q^{[1]}(x)=\sum_{s=1}^{n_M} M_s\,\delta(x-x_s)\quad,\quad q^{[2]}(x)=\sum_{k=1}^{n_M} M_s\,U(x-x_s)\quad, \tag{33}$$
$$q^{[3]}(x)=\sum_{s=1}^{n_M} M_s\,(x-x_s)U(x-x_s)\quad,\quad q^{[4]}(x)=\sum_{s=1}^{n_M} M_s\,\frac{(x-x_s)^2}{2}U(x-x_s)\quad.$$

where $\delta^{[1]}(x-x_s)$ indicates the generalized primitive function of the Dirac's delta distribution, called doublet distribution, which is able to model concentrated moments.

5.1 Parabolic flexural stiffness

For a non-uniform beam whose flexural stiffness law is parabolic, the following model is assumed:

$$E(x)I(x)=E_0I_0\left(1+k_1x+k_2x^2\right) \tag{34}$$

The flexural stiffness parameters E_0I_0, k_1 and k_2 appearing in Eqs. (34) can also be chosen to obtain a uniform beam ($E_0I_0\neq 0$, $k_1=k_2=0$) and a linear beam ($E_0I_0\neq 0, k_1\neq 0$, $k_2=0$). In this case, if the uniform distributed load q_0 is considered, the governing differential equation can be obtained by Eq. (2) as follows:

$$\left(1+k_1x+k_2x^2\right)u^{IV}(x)+2\left(k_1+2k_2x\right)u^{III}(x)+2k_2u^{II}(x)=\frac{q_0}{E_0I_0} \tag{35}$$

The solution of the governing equation (35) is obtained by replacing Eq. (34) into Eq. (11), and after calculation of the integrals, as follows:

$$\begin{aligned}u(x)&=c_1+c_2x+2c_3\left[\left(k_1+2k_2x\right)F(x)-G(x)\right]+\\&+6c_4\frac{1}{k_2}\left[\left(k_1+k_2x\right)G(x)-\left(k_1k_2x+k_1^2-2k_2\right)F(x)-x\right]+\\&+\frac{q_0}{4k_2^2E_0I_0}\left[2\left(k_1k_2x+k_1^2-k_2\right)\left(k_1F(x)-G(x)\right)-4k_2\left(k_1+k_2x\right)F(x)+\left(2k_1+k_2x\right)\right.\end{aligned} \tag{36}$$

In Eq. (36) the following functions have been defined:

$$F(x)=\frac{1}{2k_2\sqrt{k_1^2-4k_2}}\ln\left[\frac{2+\left(k_1+\sqrt{k_1^2-4k_2}\right)x}{2+\left(k_1-\sqrt{k_1^2-4k_2}\right)x}\right]$$
$$G(x)=\frac{1}{2k_2}\ln\left(1+k_1x+k_2x^2\right) \tag{37}$$

Successive differentiations of Eq. (36) leads to:

$$\varphi(x)=-u^{I}(x)=-c_2-4c_3k_2F(x)-6c_4\left(k_1F(x)-G(x)\right)-$$
$$-\frac{q_0}{2k_2E_0I_0}\left[\left(k_1^2-2k_2\right)F(x)-k_1G(x)+x\right]$$
$$M(x)=-E(x)I(x)u^{II}(x)=-E_0I_0\left(2c_3+6c_4x+\frac{q_0x^2}{2E_0I_0}\right) \tag{38}$$
$$V(x)=M^{I}(x)=-E_0I_0\left(6c_4+\frac{q_0x}{E_0I_0}\right)$$

By substituting Eqs. (38) into Eq. (35), the governing differential equation is identically satisfied.

5.2 Hyperbolic flexural stiffness

For a non-uniform beam, whose flexural stiffness is variable with an hyperbolic law, the following model is assumed:

$$E(x)I(x)=\frac{E_0I_0}{1+kx} \tag{39}$$

In this case, if a uniformly distributed load $q(x)=q_a$ is considered, the governing differential equation can be obtained by Eq. (2) as follows:

$$\frac{1}{1+kx}u^{IV}(x)-\frac{2k}{\left(1+kx\right)^2}u^{III}(x)+\frac{2k^2}{\left(1+kx\right)^3}u^{II}(x)=\frac{q_0}{E_0I_0}\left(1+\alpha_1x+\alpha_2x^2\right) \tag{40}$$

The solution of the governing equation (40) is obtained by replacing Eqs. (31) and (39) into Eq. (11), and after calculation of the integrals, as follows:

$$u(x)=c_1+c_2x+c_3\left(\frac{k}{3}x+1\right)x^2+c_4\left(\frac{k}{2}x+1\right)x^3+$$
$$+\frac{q_0}{24E_0I_0}\left(\frac{k\alpha_2}{21}x^3+\frac{2k\alpha_1+\alpha_2}{15}x^2+\frac{\alpha_1+3k}{5}x+1\right)x^4 \tag{41}$$

Successive differentiations of Eq. (41) leads to:

$$\varphi(x)=-u^{I}(x)=-c_2-c_3\left(kx+2\right)x-c_4\left(2kx+3\right)x^2-$$
$$-\frac{q_0}{24E_0I_0}\left[\frac{k\alpha_2}{3}x^3+\frac{2\left(2k\alpha_1+\alpha_2\right)}{5}x^2+\left(\alpha_1+3k\right)x+4\right]x^3 \tag{42}$$

$$M(x) = -E(x)I(x)u^{II}(x) = -E_0 I_0 \left[2c_3 + 6c_4 x + \frac{q_0}{12 E_0 I_0} \left(6 + 2\alpha_1 x + \alpha_2 x^2 \right) x^2 \right]$$

By substituting Eqs. (42) into Eq. (40), the governing differential equation is identically satisfied.

6 NUMERICAL APPLICATION TO A NON-UNIFORM BEAM WITH SINGULARITIES

The closed form solution presented in this work under the form reported in Eq.(28) allows to treat straightforwardly any non-uniform beam in presence of abrupt changes of the flexural stiffness and also singularities causing slope discontinuities, provided that the four integration constants are evaluated by means of the boundary conditions at both ends of the beam.

As a matter of example, the clamped-clamped non-uniform beam with singularities, and subjected to a transversal load $q(x) = q_0 = 40\ kN/m$, uniformly distributed along the entire span of lenght $l = 5\ m$, has been analysed according to the following flexural stiffness:

$$E(x)I(x) = E_0 I_0 \left(1 + k_1 x + k_2 x^2\right) \left[1 - \gamma U\left(x - x_\gamma\right) - \sum_{j=1}^{2} \beta_j \delta\left(x - x_{\beta,j}\right) \right] \tag{43}$$

In particular, the beam under study presents a parabolic flexural stiffness with the following values for the parameters appearing in Eq.(43): $E_0 = 2.06 \cdot 10^8\ kN/m^2$, $I_0 = 8.57 \cdot 10^{-3}\ m^4$, $k_1 = -0.4\ m^{-1}$, $k_2 = 0.1\ m^{-2}$. Moreover, the beam is subjected to an abrupt change of the flexural stiffness at $x_\gamma = 2.3\ m$ whose intensity is defined by $\gamma = 0.7$, and to two internal hinges at $x_{\beta,1} = 1.8\ m$, $x_{\beta,2} = 4.5\ m$ endowed with rotational springs whose stiffnesses are defined by the parameters $\beta_1 = 0.18\ m$, $\beta_2 = 0.11\ m$, respectively. For the quantity A the first value, $A = 2.013$ evaluated in[10], among those proposed in[11] has been chosen. The adopted flexural stiffness variation is reported in Fig.2.

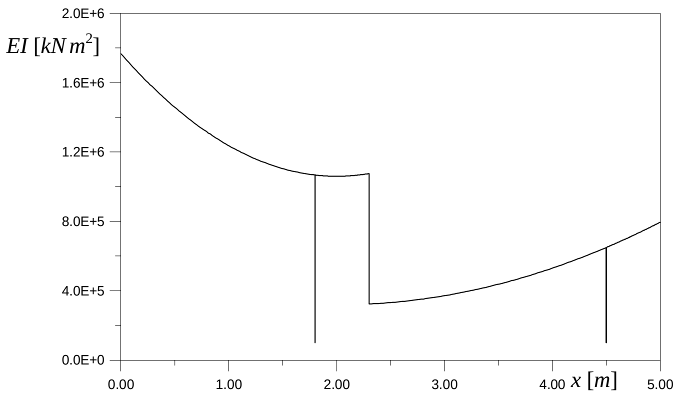

Figure 2: The adopted flexural stiffness variation

The closed form soution for the beam under study, in terms of transversal displacement, is directly inferred by Eq.(28) as follows:

$$\begin{aligned} u(x) = {} & c_1 + c_2 x + 2c_3 \left[\left(k_1 + 2k_2 x\right) F(x) - G(x) + l \sum_{j=1}^{2} \bar{\beta}_j^{(\text{lim})} \left(x - x_{\beta,j}\right) U\left(x - x_{\beta,j}\right) \right] \\ & + 6c_4 \frac{1}{k_2} \left[\left(k_1 + k_2 x\right) G(x) - \left(k_1 k_2 x + k_1^2 - 2k_2\right) F(x) - x + l \sum_{j=1}^{2} \bar{\beta}_j^{(\text{lim})} x_{\beta,j} \left(x - x_{\beta,j}\right) U\left(x - x_{\beta,j}\right) \right] \\ & + \frac{q_0}{4k_2^2 E_0 I_0} \left[2\left(k_1 k_2 x + k_1^2 - k_2\right)\left(k_1 F(x) - G(x)\right) - 4k_2 \left(k_1 + k_2 x\right) F(x) + \left(2k_1 + k_2 x\right) x \right] \\ & + l \sum_{j=1}^{2} \bar{\beta}_j^{(\text{lim})} \frac{q^{[2]}(x_{\beta,j})\left(x - x_{\beta,j}\right)}{E(0)I(0)} U\left(x - x_{\beta,j}\right) \end{aligned} \tag{44}$$

where $F(x)$, $G(x)$ are given by Eqs.(37a), (37b), respectively, and $\bar{\beta}_j^{(\text{lim})}$ is given by Eq.(29).

The integration constants c_1, c_2, c_3, c_4 appearing in Eq.(44) have been determined by imposing the following boundary conditions:

$$u(0) = 0 \quad , \quad u'(0) = 0 \quad , \quad u(l) = 0 \quad , \quad u'(l) = 0 \tag{45}$$

and the results, in terms of transversal displacement, rotation, curvature, bending moment and shear force have been plotted in Figs.3-7, respectively, and compared to those regarding a uniform beam with constant flexural stiffness $E_0 I_0$ and a non-uniform beam with parabolic flexural stiffness $E(x)I(x) = E_0 I_0 \left(1 + k_1 x + k_2 x^2\right)$ in absence of singularity of any kind.

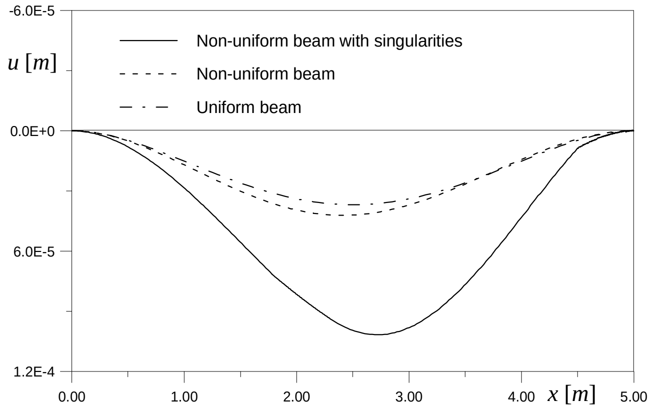

Figure 3: Transversal displacements

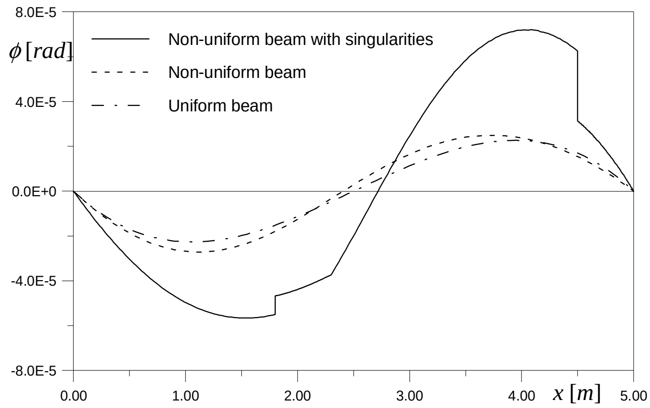

Figure 4: Rotation function

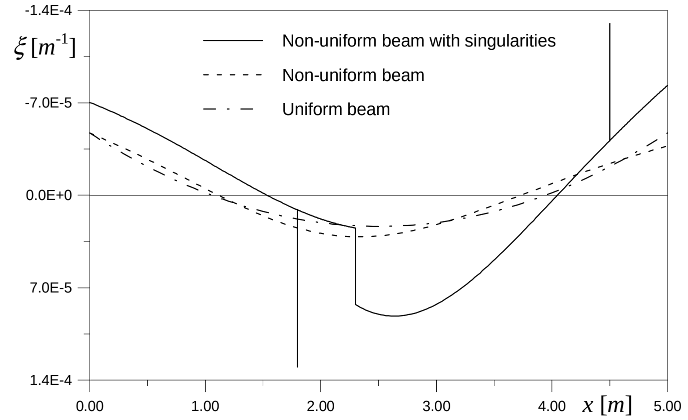

Figure 5: Curvature function

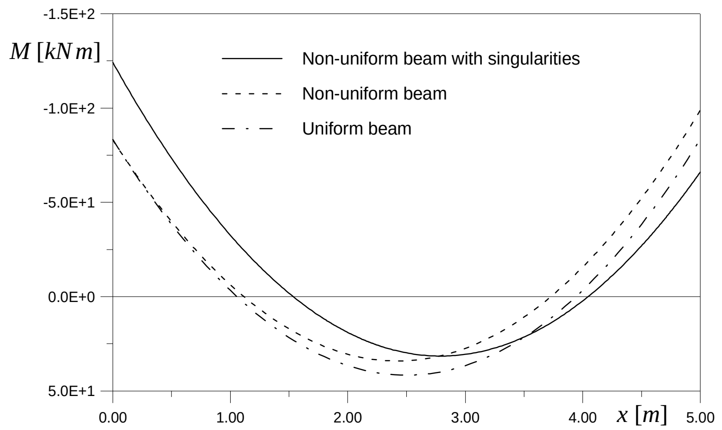

Figure 6: Bending moment

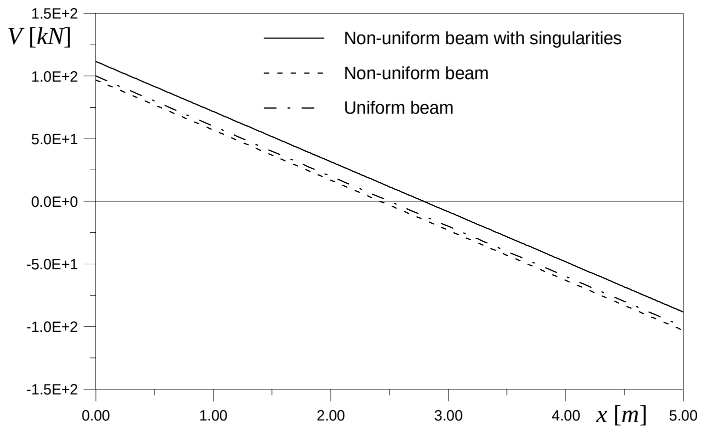

Figure 7: Shear force

7 NON-UNIFORM BEAMS DUE TO THE PRESENCE OF DAMAGE

Variations of geometrical and structural parameters along the beam span are often caused by the presence of both concentrated and diffused damage. In particular, the presence of concentrated cracks along the beam span can be shown to lead to a stiffness reduction in a beam segment of finite width, whose extension and intensity can be obtained by means of the theory of fracture mechanics or from a definition of the "ineffective area" so as to define a distributed damage model[13-18]. The models for the concentrated cracks, as long as damage evolution is not accounted for, can hence be treated within the context of the theory of beams with non-uniform geometrical and physical parameters. The theory and the exact closed form solutions presented in this work can hence be adopted to analyse some of the different damage models available in the literature.

The effect of a single crack of depth d_c, concentrated at the abscissa x_c of the beam, can be represented by a variable flexural stiffness $E(x)I(x)$ as follows:

$$E(x)I(x) = E_0 I_0 \, g(x) \tag{46}$$

where the $g(x)$ function is able to describe the flexural stiffness decrement in the vicinity of the crack. In this context, the influence of a crack on a rectangular cross-section of height d, by neglecting the change of the neutral axis in the vicinity of the crack, will be considered. In particular, the ratio of the flexural stiffness value $E_c I_c$ at the cracked cross-section to the undamaged value $E_0 I_0$ will be denoted as $g_c = (d - d_c)^3 / d^3$. Furthermore, L_c denotes the so-called "effective length" accounting for the influence of the cracked cross-section on the flexural stiffness of the beam.

The damage models provided in the literature differ from one another for the expression of the $g(x)$ function and for the amplitude of the effective length L_c.

Cerri and Vestroni[13,14] proposed a uniform variation of the flexural stiffness provided as follows:

$$g(x) = \begin{cases} 1 & \text{for} \quad x \le x_c - L_c \\ g_c & \text{for} \quad x_c - L_c < x < x_c + L_c \\ 1 & \text{for} \quad x \ge x_c + L_c \end{cases} \tag{47}$$

where the effective length has been considered independent of the crack depth and assumed as $L_c = 1.5\,d$.

Sinha et al.[17] proposed a linear variation of the flexural stiffness provided as follows:

$$g(x) = \begin{cases} 1 & \text{for} \quad x \le x_c - L_c \\ g_c + (1 - g_c) \cdot \dfrac{|x - x_c|}{L_c} & \text{for} \quad x_c - L_c < x < x_c + L_c \\ 1 & \text{for} \quad x \ge x_c + L_c \end{cases} \tag{48}$$

where the effective length is evaluated as follows: $L_c = \dfrac{d}{\alpha} \cdot \dfrac{\ln g_c}{g_c - 1}$ with $\alpha = 0.667$.

Bilello[18], on the basis of photoelastic analysis, confirmed by numerical finite element analysis, considered a triangular shaped "ineffective area" in the vicinity of the crack, subjected to very low stress, leading to a cubic variation of the flexural stiffness provided as follows:

$$g(x)=\begin{cases}1 & \text{for } x\le x_c-L_c\\ \left(\sqrt[3]{g_c}+\left(1-\sqrt[3]{g_c}\right)\cdot\dfrac{|x-x_c|}{L_c}\right)^3 & \text{for } x_c-L_c<x<x_c+L_c\\ 1 & \text{for } x\ge x_c+L_c\end{cases} \tag{49}$$

where the effective length is given as $L_c=\dfrac{d_c}{0.9}$.

An exponential variation of the flexural stiffness due to the presence of a crack has been proposed by Christides and Barr[15] and Shen and Pierre[16] as follows:

$$g(x)=\frac{g_c}{g_c+\left(1-g_c\right)\cdot\exp\left(-2\alpha\dfrac{|x-x_c|}{d}\right)} \tag{50}$$

where no effective length has been introduced and the value of the parameter α, ruling the exponential law, has been obtained by experimental investigation as $\alpha=0.667$ by Christides and Barr[15] and $\alpha=1.936$ by Shen and Pierre[16].

The damage models reported in this section can be treated in the context of the non-uniform beam analysis. The solution proposed in this work for non-uniform beams can hence be adopted in order to provide a comparison regarding the different damage models in terms of resulting transversal displacements.

A simply supported beam with length $L=1800\,mm$ and rectangular cross-section (width $b=50\,mm$ height $d=25\,mm$) in presence of a single crack concentrated at $x_c=1200\,mm$ with two different values of crack depth $d_c=6.25\,mm, 12.5\,mm$ has been considered. In Fig.8 the different damage models reported in this section have been adopted to treat the case of crack depth $d_c=6.25\,mm$ correspondent to $25\,\%$ damage percentage. In particular in Fig.8a the different flexural stiffness distributions have been depicted; in Fig.8b the relevant transversal displacement, normalised with respect to the maximum displacement u_0 of the undamaged beam, showing small differences, have been plotted.

In Fig.9 the results concerning the case of crack depth $d_c=12.5\,mm$ correspondent to $50\,\%$ damage percentage are reported. In particular in Fig.9a and 9b the different flexural stiffness distributions and the relevant transversal displacement, normalised with respect to the maximum displament u_0 of the undamaged beam, respectively, have been plotted. According to Figs.8b and 9b, the results provided by the different damage models show differences increasing with the damage intensity.

8 CONCLUSIONS

The problem of integration of the static governing equations of non-uniform Euler-Bernoulli beams has been treated in this study. The general explicit solution, requiring the evaluation of four integration constants dependent on the boundary conditions, has been presented. The

solution has been obtained as the limit of the case of multi-stepped beams, as a consequence the solution has been extended to non-uniform beam in presence of abrupt changes of the flexural stiffness. Furthermore, singularities due to the presence of Dirac's delta distributions has been also introduced in the model to treat those cases in which internal hinges with rotational springs are present along the beam span. In any case, the proposed integration procedure leads to general closed form solutions without enforcement of any continuity condition along the beam span and only four integration constants are to be determined.
The presented closed form solutions are adopted to provide explicit expressions for different non-uniform beams with different external load functions. A numerical application to show the efficiency of the proposed solutions to a non-uniform beam with a non-differentiable flexural stiffness due to the presence of different singularities has been presented. Finally, the case of a damaged beam subjected to concentrated damages has been treated within the context of non-uniform beam analysis. In fact, according to the damage models available in the literature, the effect of concentrated cracks can be treated as a non-uniform flexural stiffness distribution.

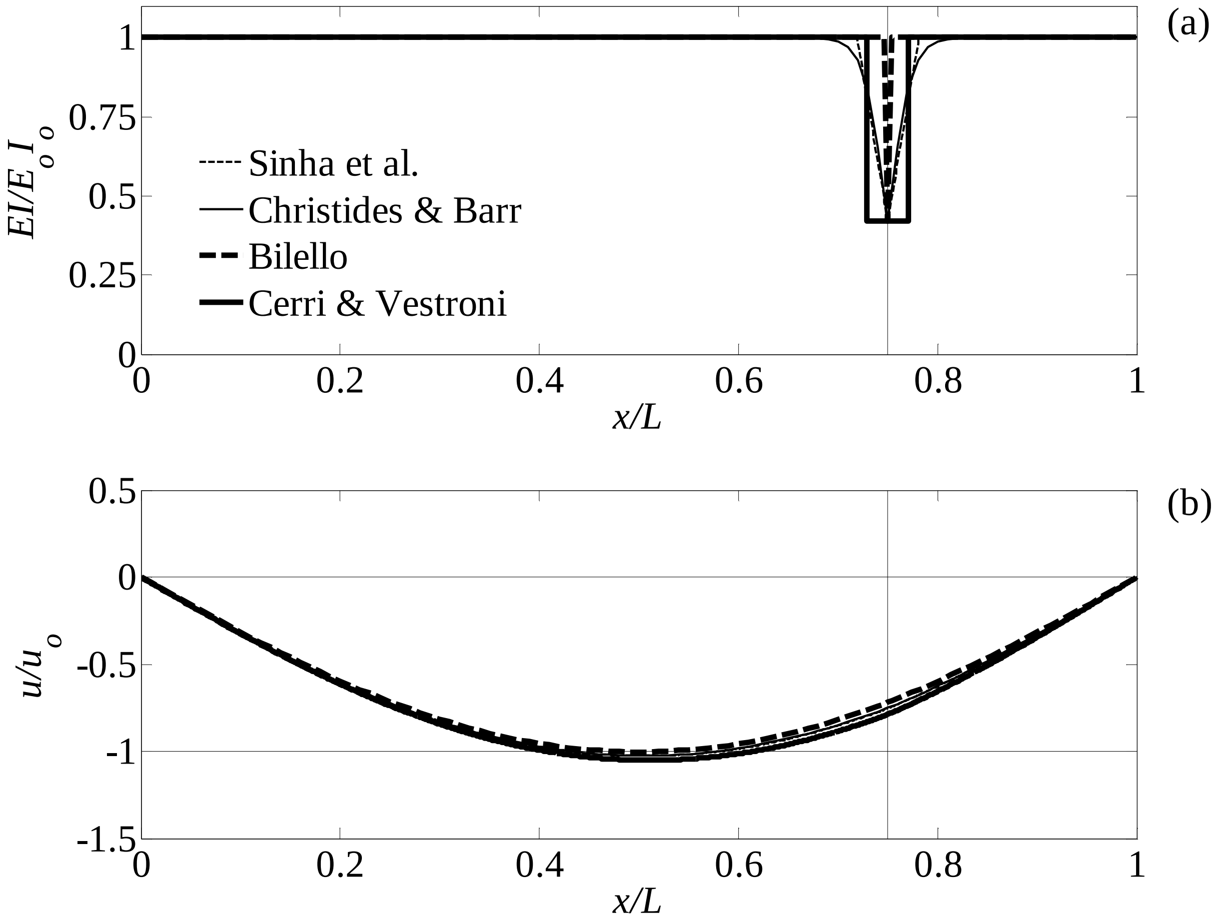

Figure 8: Beam with a concentrated crack correspondent to 25 % damage percentage: a) flexural stiffness distribution; b) transversal displacements

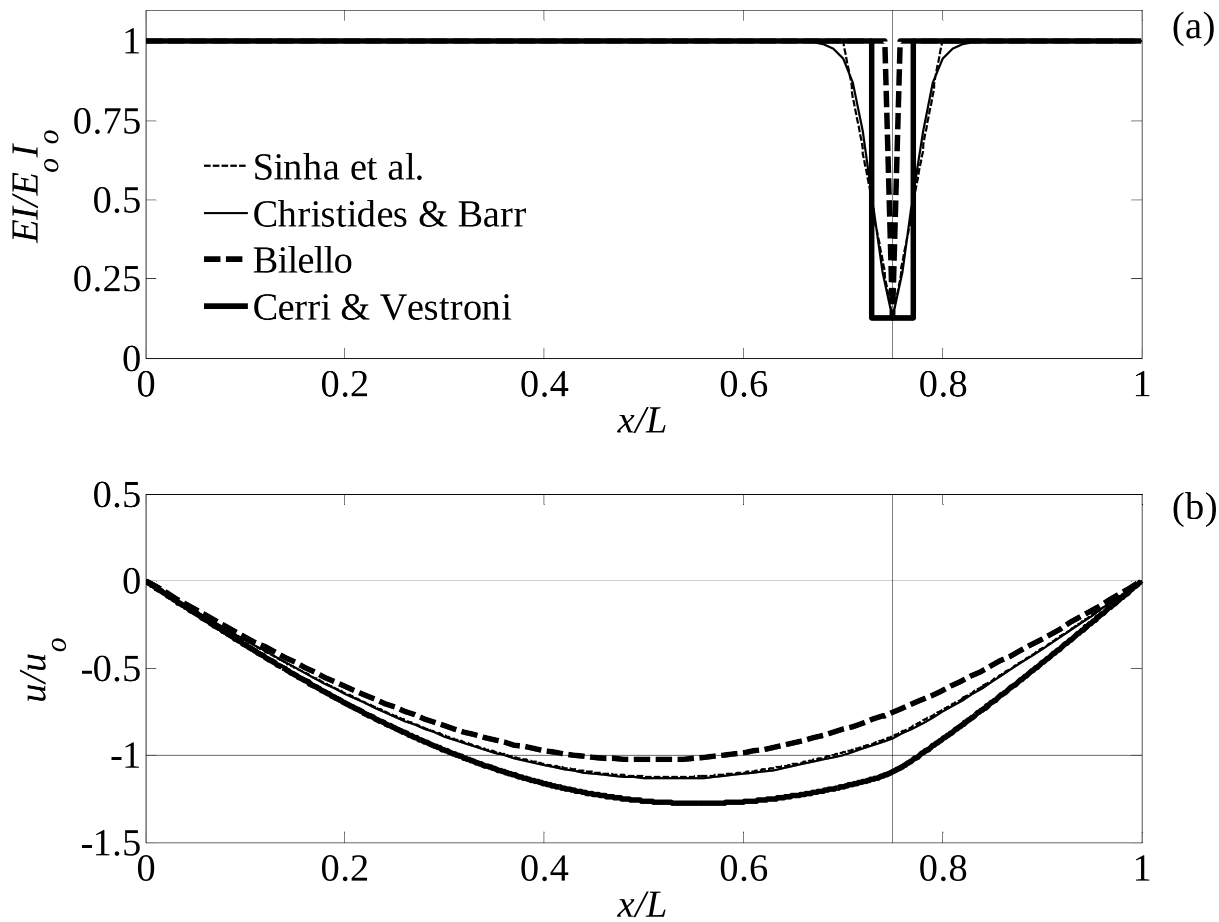

Figure 9: Beam with a concentrated crack correspondent to 50 % damage percentage: a) flexural stiffness distribution; b) transversal displacements

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