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INTERVAL VERSUS STOCHASTIC FINITE ELEMENT ANALYSIS OF STRUCTURES WITH UNCERTAIN PARAMETERS

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Abstract. *In order to handle uncertainties, which unavoidably affect structural analysis, traditionally the probability theory has been an essential cornerstone. Nevertheless, recently criticism has arisen towards the effectiveness of building a Probability Density Function to characterize the uncertain properties when experimental data are not sufficient to justify it. In this context, the interval model has proved to be a useful tool to face the lack of knowledge characterizing early stages of the design process. The aim of this paper is to compare structural responses obtained by applying the interval and stochastic approaches to incorporate uncertainties into finite element analysis.*

1 INTRODUCTION

In engineering practice, it is customary to employ deterministic values for the design parameters, relying on the concept of “safety factors” to take into account the sources of uncertainty which may affect all the stages of the design process. The main drawback arising from this *modus operandi* is that the choice of conservative values for those safety factors, although essential, may lead to an overestimation of the dimensions of the structural components. Over the last decades, huge efforts have been devoted to incorporate uncertainties within numerical analyses as actual design parameters, in order to obtain more realistic models of physical phenomena.

Traditionally, uncertainties have been modeled in the context of the classical probability theory as random variables or random fields. Nevertheless, sometimes the inadequacy of experimental data to define the Probability Density Function (PDF) of the uncertain properties does not justify the use of the probabilistic model, as it often occurs in early stages of the design process. For this reason, recently a growing interest has arisen towards alternative approaches based on non-probabilistic concepts. In this context, the interval model, originally developed from the *Classical Interval Analysis* (*CIA*)[1], has gained popularity because it only requires the knowledge of the range of variability of the uncertain parameters, which are indeed modeled as interval variables with given lower bound (LB) and upper bound (UB).

Over the last decades, researchers' efforts have been devoted to incorporate uncertainties into the Finite Element Method, which nowadays represents the most efficient tool to analyze complex engineering systems. As a result, several Interval Finite Element Methods (IFEMs)[2] and Stochastic Finite Element Methods (SFEMs)[3] have been developed. In particular, within the interval framework, a new IFEM[4] based on the *improved interval analysis* via *extra unitary interval* (*IIA* via *EUI*)[5] has been recently introduced by the authors. This method enables to overcome the main drawbacks that hinder the use of IFEMs in engineering applications, such as for instance their inability to handle relatively high degrees of uncertainty, or to find accurate estimates of stress bounds.

In this paper, the IFEM based on the *IIA* is exploited to perform appropriate comparisons with the results provided by the traditional probabilistic model in the context of finite element analysis. To this aim, the uncertain properties are consistently modeled as random variables uniformly distributed within the range of the corresponding interval variables. Approximate closed-form expressions of the mean-value and variance of the stochastic response are derived by applying the so-called *Rational Series Expansion* (*RSE*)[6], recently introduced to approximate the explicit inverse of a matrix with small rank-*r* modifications.

A numerical application concerning a square plate under uniform traction with uncertain Young's modulus is presented to compare the structural responses pertaining to the interval and stochastic models of uncertainties.

2 FINITE ELEMENT FORMULATION FOR STRUCTURES WITH UNCERTAIN PROPERTIES

Let us consider a continuous body made of linear-elastic isotropic material which occupies the volume V bounded by the surface S in its undeformed state. The body is subjected to volume forces $\mathbf{b}(\mathbf{x})$ in V and surface forces $\mathbf{t}(\mathbf{x})$ on the portion S_t of the boundary surface S, with $\mathbf{x}=[x_1 \quad x_2 \quad x_3]^{\mathrm{T}}$ denoting the position vector of a generic point referred to a Cartesian coordinate system $O(x_1,x_2,x_3)$; the displacements $\tilde{\mathbf{u}}(\mathbf{x})$ are imposed on the constrained portion S_u of S, such that $S=S_t \cup S_u$. Applied loads are assumed to be deterministic and to act in a quasi-static manner. Let the volume of the body be subdivided into N_e Finite Elements (FEs). Without loss of generality, only Young's modulus of the material is treated as an uncertain parameter, while all other input parameters are supposed to be deterministic. In particular, the present formulation relies on the assumption of independent uncertain Young's moduli of the FEs, defined as follows:

$$E^{(i)}(\alpha_i)=E_0^{(i)}\left(1+\alpha_i\right), \qquad (i=1,2,\dots,N_e) \tag{1}$$

where $\alpha_i<1$ is the dimensionless fluctuation around the nominal value $E_0^{(i)}$. At this stage, no assumptions are introduced on the uncertainty model assumed for the fluctuations α_i .

Relying on Eq.(1), the elastic matrix of the i-th FE reads as:

$$\mathbf{E}^{(i)}(\alpha_i)=\left(1+\alpha_i\right)\mathbf{E}_0^{(i)} \tag{2}$$

where $\mathbf{E}_0^{(i)}$ is the nominal value. Following the standard displacement-based FE formulation, the displacement field, $\mathbf{u}^{(i)}(\mathbf{x};\boldsymbol{\alpha})$, the strain field, $\boldsymbol{\varepsilon}^{(i)}(\mathbf{x};\boldsymbol{\alpha})$ and the stress field, $\boldsymbol{\sigma}^{(i)}(\mathbf{x};\boldsymbol{\alpha})$, within the i-th FE are expressed as interpolation of the nodal displacements collected into the vector $\mathbf{d}^{(i)}(\boldsymbol{\alpha})$, i.e.:

$$\begin{aligned}&\mathbf{u}^{(i)}(\mathbf{x};\boldsymbol{\alpha})=\mathbf{N}^{(i)}(\mathbf{x})\mathbf{d}^{(i)}(\boldsymbol{\alpha});\\&\boldsymbol{\varepsilon}^{(i)}(\mathbf{x};\boldsymbol{\alpha})=\mathbf{B}^{(i)}(\mathbf{x})\mathbf{d}^{(i)}(\boldsymbol{\alpha});\\&\boldsymbol{\sigma}^{(i)}(\mathbf{x};\boldsymbol{\alpha})=\mathbf{E}^{(i)}(\alpha_i)\mathbf{B}^{(i)}(\mathbf{x})\mathbf{d}^{(i)}(\boldsymbol{\alpha})\end{aligned} \tag{3a-c}$$

where $\boldsymbol{\alpha}$ is the vector collecting the dimensionless fluctuations of Young's moduli α_i $(i=1,2,\ldots,N_e)$. In the previous equations, $\mathbf{N}^{(i)}(\mathbf{x})$ is the shape-function matrix and $\mathbf{B}^{(i)}(\mathbf{x})$ is the strain-displacement matrix.

The stiffness matrix of the *i*-th FE is formally analogous to the one pertaining to the deterministic FE and it can be expressed as the result of a fluctuation around the nominal stiffness matrix $\mathbf{k}_0^{(i)}=\mathbf{k}^{(i)}(\alpha_i)\big|_{\boldsymbol{\alpha}=\mathbf{0}}$, as follows:

$$\mathbf{k}^{(i)}(\alpha_i)=\int_{V^{(i)}}\mathbf{B}^{(i)\mathrm{T}}(\mathbf{x})\mathbf{E}^{(i)}(\alpha_i)\mathbf{B}^{(i)}(\mathbf{x})\mathrm{d}V^{(i)}=\left(1+\alpha_i\right)\mathbf{k}_0^{(i)}. \tag{4}$$

Furthermore, the hypothesis of deterministic applied loads entails that uncertainties do not affect the element force vector, i.e.:

$$\mathbf{f}^{(i)}=\int_{V^{(i)}}\mathbf{N}^{(i)\mathrm{T}}(\mathbf{x})\mathbf{b}(\mathbf{x})\mathrm{d}V^{(i)}+\int_{S_t^{(i)}}\mathbf{N}^{(i)\mathrm{T}}(\mathbf{x})\mathbf{t}(\mathbf{x})\mathrm{d}S^{(i)}. \tag{5}$$

By performing standard assembly procedure, the set of linear equilibrium equations is obtained:

$$\mathbf{K}(\boldsymbol{\alpha})\mathbf{U}(\boldsymbol{\alpha})=\mathbf{F}. \tag{6}$$

In Eq. (6), $\mathbf{U}(\boldsymbol{\alpha})$ is the $n-$vector of the unknown global displacements, n being the number of degrees of freedom of the FE model, while

$$\mathbf{K}(\boldsymbol{\alpha})=\sum_{i=1}^{N_e}\mathbf{L}^{(i)\mathrm{T}}\mathbf{k}^{(i)}(\alpha_i)\mathbf{L}^{(i)}=\mathbf{K}_0+\sum_{i=1}^{N_e}\mathbf{L}^{(i)\mathrm{T}}\mathbf{k}_0^{(i)}\mathbf{L}^{(i)}\alpha_i \tag{7}$$

and

$$\mathbf{F}=\sum_{i=1}^{N_e}\mathbf{L}^{(i)\mathrm{T}}\mathbf{f}^{(i)} \tag{8}$$

are the global stiffness matrix and the nodal force vector, respectively, with $\mathbf{L}^{(i)}$ denoting the connectivity matrix. Notice that the global stiffness matrix $\mathbf{K}(\boldsymbol{\alpha})$, which depends on the fluctuations α_i, can be expressed as the sum of the global nominal stiffness matrix $\mathbf{K}_0$ plus a deviation given by the superposition of the contributions of each uncertain parameter.

Whatever uncertainty model is adopted, the knowledge of the inverse of the global stiffness matrix as an explicit function of the uncertain parameters plays a crucial role in order to predict the variability of the response. In this context, recently the so-called *Rational Series*

Expansion (*RSE*)[6] has been introduced to approximate the explicit inverse of a matrix with small rank-*r* modifications. The first step to apply the *RSE* is the decomposition of the stiffness matrix as sum of the nominal value plus a deviation given by a superposition of rank-one matrices. For this purpose, the following decomposition is herein applied:

$$\mathbf{K}(\boldsymbol{\alpha}) = \mathbf{K}_0 + \sum_{i=1}^{N_e}\sum_{\ell=1}^{p_i} \lambda_i^{(\ell)} \mathbf{v}_i^{(\ell)} \mathbf{v}_i^{(\ell)\mathrm{T}} \alpha_i \,; \quad \mathbf{v}_i^{(\ell)} = \mathbf{L}^{(i)\mathrm{T}} \boldsymbol{\phi}_i^{(\ell)} \tag{9a,b}$$

where $\lambda_i^{(\ell)}$ and $\boldsymbol{\phi}_i^{(\ell)}$ denote the ℓ-th eigenvalue and the associated eigenvector of the nominal stiffness matrix $\mathbf{k}_0^{(i)}$ of the i-th FE. It is worth emphasizing that the number of non-zero eigenvalues, $p_i < n$, coincides with the number of deformation modes of the *i*-th FE. If the degree of uncertainty is small, namely $\alpha_i << 1$, only first-order terms of the *RSE* can be retained, yielding the following approximate explicit expression of the inverse of the stiffness matrix (*RSE*-1):

$$\mathbf{K}(\boldsymbol{\alpha})^{-1} \approx \mathbf{K}_0^{-1} - \sum_{i=1}^{N_e}\sum_{\ell=1}^{p_i} \frac{\alpha_i \lambda_i^{(\ell)}}{1+\lambda_i^{(\ell)} d_{i\ell}\, \alpha_i} \mathbf{D}_{i\ell} \tag{10}$$

where

$$d_{i\ell} = \mathbf{v}_i^{(\ell)\mathrm{T}} \mathbf{K}_0^{-1} \mathbf{v}_i^{(\ell)}; \quad \mathbf{D}_{i\ell} = \mathbf{K}_0^{-1} \mathbf{v}_i^{(\ell)} \mathbf{v}_i^{(\ell)\mathrm{T}} \mathbf{K}_0^{-1}. \tag{11a,b}$$

Equation (10) holds if and only if $\left|\lambda_i^{(\ell)} \alpha_i d_{i\ell}\right| < 1$.

3 UNCERTAIN PARAMETERS MODELED AS INTERVAL VARIABLES

The FE formulation presented in the previous section holds regardless of the model assumed to describe the uncertain parameters. The aim of the present study is to compare the stochastic and interval models of uncertainty by examining the associated structural responses.

First, the uncertain parameters are modeled as independent interval variables in the context of the *improved interval analysis* via *extra unitary interval* (*IIA* via *EUI*)[5], recently introduced in the literature to limit the conservatism due to the so-called *dependency phenomenon*[1] which arises when the same interval variable occurs more than once in a mathematical expression. According to the *IIA*[5], the fluctuation of Young's modulus of the *i*-th FE can be modeled as a symmetric interval variable defined as:

$$\alpha_i^I = \alpha_{0,i} + \Delta\alpha_i \hat{e}_i^I = \Delta\alpha_i \hat{e}_i^I \tag{12}$$

where the apex *I* characterizes the interval variables; $\alpha_{0,i}$ is the midpoint value (or mean) and $\Delta\alpha_i$ is the deviation amplitude (or radius), given by:

$$\alpha_{0,i} = \frac{\underline{\alpha}_i + \overline{\alpha}_i}{2} = 0; \quad \Delta\alpha_i = \frac{\overline{\alpha}_i - \underline{\alpha}_i}{2} > 0 \tag{13a,b}$$

with the symbols $\underline{\alpha}_i$ and $\overline{\alpha}_i$ denoting the lower bound (LB) and upper bound (UB) of the interval, respectively. Notice that the conditions $\Delta\alpha_i < 1$ need to be satisfied in order to ensure positive values of Young's moduli. Furthermore, in Eq.(12), $\hat{e}_i^I = [-1,1]$ is the so-called *EUI* which does not follow the rules of the *Classical Interval Analysis* (*CIA*). The subscript i means that the *EUI* is associated to the *i*-th interval variable. In this way, the IFEM based on the *IIA*[4] is able to take into account the dependencies between interval variables and thus to limit the overestimation of the response due to the *dependency phenomenon*. In the interval framework, the global equilibrium equations in Eq.(6) become interval equations, i.e.:

$$\mathbf{K}^I\mathbf{U}^I = \mathbf{F}. \tag{14}$$

The solution set of these equations is typically described by a complicated region in the output space. For this reason, it is customary to seek, for each component of the interval displacement vector $\mathbf{U}^I$, the narrowest interval containing the set of all possible solutions. The interval extension of the *RSE*-1 (Eq.(10)), i.e. the *IRSE*-1[4], allows one to derive an approximate explicit expression of the interval displacement vector $\mathbf{U}^I$:

$$\mathbf{U}^I = \left(\mathbf{K}^I\right)^{-1}\mathbf{F} = \mathbf{K}_0^{-1}\mathbf{F} - \sum_{i=1}^{N_e}\sum_{\ell=1}^{p_i}\frac{\Delta\alpha_i\hat{e}_i^I\lambda_i^{(\ell)}}{1+\lambda_i^{(\ell)}d_{i\ell}\,\Delta\alpha_i\hat{e}_i^I}\mathbf{D}_{i\ell}\mathbf{F} \tag{15}$$

which yields the following closed-form formulas for the LB and UB

$$\underline{\mathbf{U}}(\boldsymbol{\alpha}) = \mathrm{mid}\left\{\mathbf{U}^I\right\} - \Delta\mathbf{U}(\boldsymbol{\alpha}); \qquad \overline{\mathbf{U}}(\boldsymbol{\alpha}) = \mathrm{mid}\left\{\mathbf{U}^I\right\} + \Delta\mathbf{U}(\boldsymbol{\alpha}) \tag{16a,b}$$

where

$$\mathrm{mid}\left\{\mathbf{U}^I\right\} = \mathbf{K}_0^{-1}\mathbf{F} + \sum_{i=1}^{N_e}\sum_{\ell=1}^{p_i}a_{0,i\ell}\mathbf{D}_{i\ell}\mathbf{F}; \qquad \Delta\mathbf{U}(\boldsymbol{\alpha}) = \sum_{i=1}^{N_e}\left|\sum_{\ell=1}^{p_i}\Delta a_{i\ell}\mathbf{D}_{i\ell}\mathbf{F}\right| \tag{17a,b}$$

where the symbol $|\bullet|$ denotes absolute value component wise. In the previous equations, $a_{0,i\ell}$ and $\Delta a_{i\ell}$ are the midpoint and deviation amplitude of the generic series term in Eq.(15), respectively, i.e.:

$$a_{0,i\ell} = \frac{\left(\lambda_i^{(\ell)}\Delta\alpha_i\right)^2 d_{i\ell}}{1-\left(\lambda_i^{(\ell)}\Delta\alpha_i d_{i\ell}\right)^2}; \qquad \Delta a_{i\ell} = \frac{\lambda_i^{(\ell)}\Delta\alpha_i}{1-\left(\lambda_i^{(\ell)}\Delta\alpha_i d_{i\ell}\right)^2}. \tag{18a,b}$$

4 UNCERTAIN PARAMETERS MODELED AS RANDOM VARIABLES

Within the stochastic framework, the fluctuations α_i of the uncertain Young's moduli are modeled as independent zero-mean random variables. In order to carry out consistent comparisons with the interval model, such random variables are assumed to be uniformly distributed within the intervals $[-\Delta\alpha_i, \Delta\alpha_i]$, $\Delta\alpha_i$ being the deviation amplitude of the corresponding interval variables $\alpha_i^I = \Delta\alpha_i\hat{e}_i^I$. Under this assumption, the equilibrium equations (6) have a random nature. The probabilistic characterization of the random displacement vector $\mathbf{U}(\boldsymbol{\alpha})$ can be performed by applying classical Monte Carlo simulation (*MCS*) which requires heavy computations. Alternatively, by applying the *RSE*-1, an approximate explicit expression of the random displacement vector $\mathbf{U}(\boldsymbol{\alpha})$ can be derived

$$\mathbf{U}(\boldsymbol{\alpha}) = \mathbf{K}(\boldsymbol{\alpha})^{-1}\mathbf{F} = \mathbf{K}_0^{-1}\mathbf{F} - \sum_{i=1}^{N_e}\sum_{\ell=1}^{p_i}\frac{\alpha_i\lambda_i^{(\ell)}}{1+\lambda_i^{(\ell)}d_{i\ell}\,\alpha_i}\mathbf{D}_{i\ell}\mathbf{F} \tag{19}$$

which yields the following closed-form formulas for the mean-value and covariance matrix of $\mathbf{U}(\boldsymbol{\alpha})$:

$$\boldsymbol{\mu}_{\mathbf{U}} = \mathrm{E}\langle \mathbf{U}(\boldsymbol{\alpha})\rangle = \mathbf{U}_0 - \sum_{i=1}^{N_e}\sum_{\ell=1}^{p_i} \mathrm{E}\langle \chi_{i\ell}\rangle \mathbf{D}_{i\ell}\mathbf{F};$$

$$\boldsymbol{\Sigma}_{\mathbf{U}} = \mathrm{E}\langle \mathbf{U}(\boldsymbol{\alpha})\mathbf{U}^{\mathrm{T}}(\boldsymbol{\alpha})\rangle - \boldsymbol{\mu}_{\mathbf{U}}\left(\boldsymbol{\mu}_{\mathbf{U}}\right)^{\mathrm{T}} = \sum_{i=1}^{N_e}\sum_{\ell=1}^{p_i}\sum_{k=1}^{p_j}\left[\mathrm{E}\langle \chi_{i\ell}\chi_{ik}\rangle - \mathrm{E}\langle \chi_{i\ell}\rangle \mathrm{E}\langle \chi_{ik}\rangle\right]\mathbf{D}_{i\ell}\mathbf{F}\mathbf{F}^{\mathrm{T}}\mathbf{D}_{ik}^{\mathrm{T}} \quad (20a,b)$$

where $\mathrm{E}\langle\bullet\rangle$ denotes the stochastic average operator, while $\chi_{i\ell}$ are random variables defined as:

$$\chi_{ij} = \frac{\alpha_i \lambda_i^{(j)}}{1+\lambda_i^{(j)} d_{ij}\, \alpha_i}, \quad (j = \ell, m). \quad (21)$$

It is worth remarking that Eqs.(20a,b) are much more advantageous than classical *MCS* since they just involve the evaluation of the statistics of the random variables $\chi_{i\ell}$, without requiring the repeated inversion of the stochastic stiffness matrix $\mathbf{K}(\boldsymbol{\alpha})$.

5 NUMERICAL APPLICATION

In order to investigate the influence of the uncertainty model on structural response, the square plate with uncertain Young's modulus under uniform traction, shown in Figure 1, is analyzed. The plate is discretized into $N_e = 16$ four-node quadrilateral FEs with 8 DOFs. The following data are assumed: width and thickness of the plate $L = 0.1$ m and $t = 0.001$ m, respectively; nominal Young's modulus $E_0 = 210$ GPa and Poisson ratio $\nu = 0.3$; traction $p = 10$ MPa. The fluctuations of the uncertain Young's moduli of the FEs are modeled both as interval variables, $\alpha_i^I = \Delta\alpha_i \hat{e}_i^I$, $\Delta\alpha_i = \Delta\alpha$ ($i = 1, 2, \ldots, N_e$), and independent zero-mean random variables uniformly distributed within the interval $[-\Delta\alpha, \Delta\alpha]$.

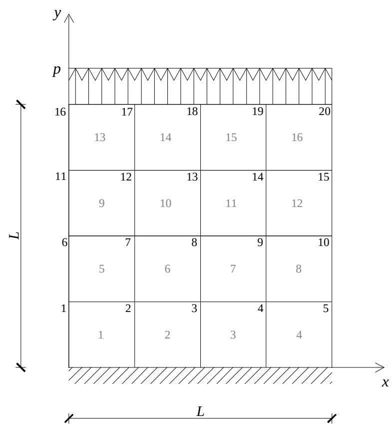

Figure 1: Square plate under uniform traction with uncertain Young's modulus.

First, the accuracy of the proposed closed-form expressions of response statistics (20a,b), obtained by applying the *RSE*-1, is scrutinized. Figure 2 displays the mean-value and variance of the nodal displacements in the load direction U_j ($j=1,2,\ldots,20$) of the plate with random Young's moduli for $\Delta\alpha = 0.2$. The comparison with the results provided by *MCS* ($N = 10000$ samples) shows the accuracy of the proposed estimates of response statistics.

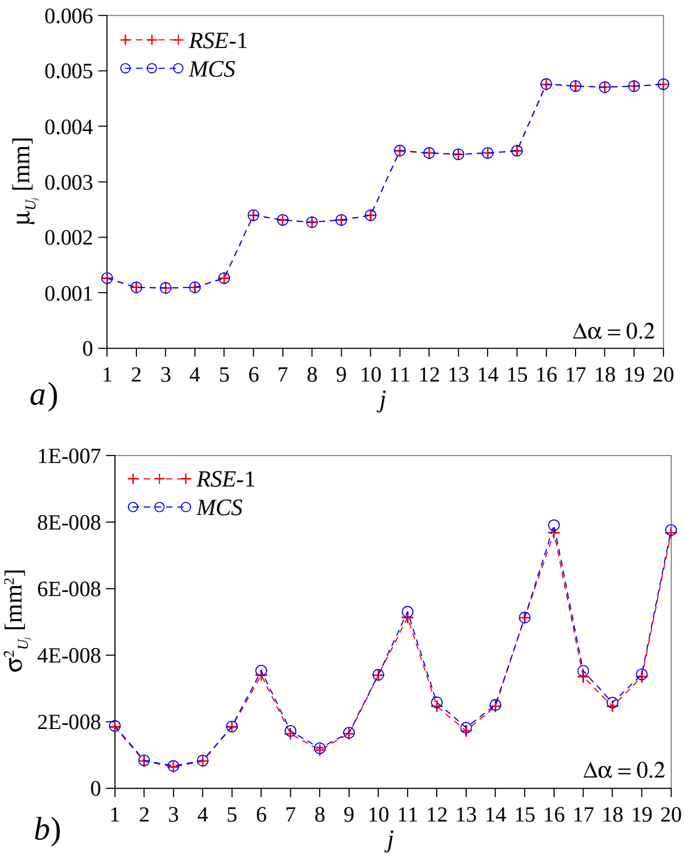

Figure 2: a) Mean-value and b) variance of the nodal displacements in the load direction of the plate with random Young's moduli: comparison between the proposed estimates (Eqs. (20a,b)) and *MCS* data ($\Delta\alpha = 0.2$).

Figure 3 displays the comparison between the region of the interval nodal displacements in the load direction of the plate provided by the *IRSE*-1 (Eqs. 16(a,b)), for $\Delta\alpha = 0.1$ and $\Delta\alpha = 0.2$, and the confidence interval $\mu_{U_j} \pm 3\sigma_{U_j}$ of the corresponding random displacements obtained by applying Eqs. (20a,b). It can be seen that the interval approach generally yields more conservative regions of the response quantities compared to those obtained within a probabilistic context.

CONCLUSIONS

The interval and stochastic approaches to handle uncertainties in structural problems have been compared in the context of FE analysis. To carry out consistent comparisons, within the stochastic framework, the uncertain properties have been modeled as independent random variables uniformly distributed over the range of the corresponding interval variables. By applying the so-called *Rational Series Expansion*, approximate explicit expressions of the bounds of the interval response and of the statistics of the random response have been derived. Numerical results, concerning a square plate with uncertain Young's modulus, have shown that the range of the interval response is generally more conservative than the confidence interval of the response given by the mean-value minus/plus three times the standard deviation.

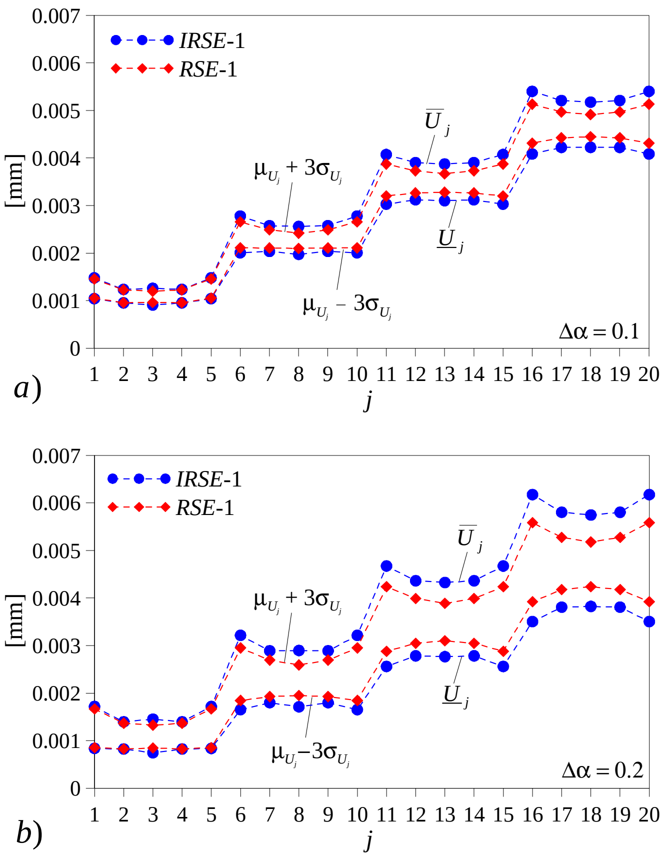

Figure 3: Comparison between the regions of the nodal displacements in the load direction of the plate with uncertain Young's moduli provided by the interval and stochastic approaches for a) $\Delta\alpha = 0.1$ and b) $\Delta\alpha = 0.2$.

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